MATH 441

Differential Equations

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Contents

1 Introduction

1.1 Some Basic Models and Direction Fields

Many laws and principles that govern the natural world are relations involving changing quantities and the rates at which they change. For example, the net force on an object as it falls is modelled by

$$m\frac{dv}{dt} = mg - \gamma v \tag{1.1}$$

where m is the mass of the object, g is the acceleration due to gravity, and γ is a damping constant.

Definition 1.1 (Differential Equation)**.** Differential equations are equations that contain the derivatives of some quantity and possibly the quantity itself.

The solution to a differential equation is a function representing the quantity involved in the derivative.

Definition 1.2 (Direction Field)**.** Direction fields are visual tools used to study solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y) \tag{1.2}$$

where f is commonly known as the *rate function.*

A direction field is constructed by evaluating f at each t and y; the resulting value is the slope of y at that t and y. This slope is the slope of the solution curve $y(t)$. To construct mathematical models using differential equations, we follow the steps:

1. Identify the independent (generally time) and dependent variable.

2. Assign units of measurement for each variable.

3. Articulate the basic principle or law that governs the problem you are investigating, like Newton's law of motion.

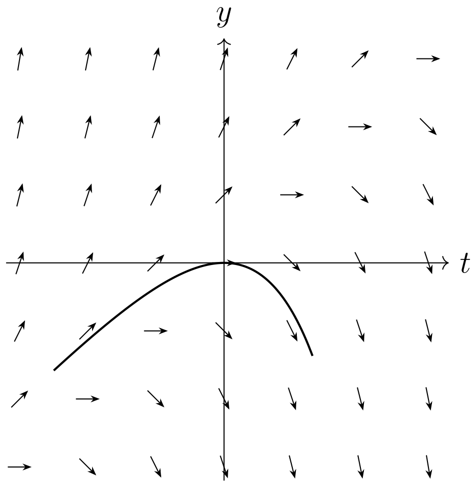


Figure 1: A direction field for $dy/dt = y - t$ with one integral curve superimposed.

4. Express the law in step 3 with variables in step 1; you may have to introduce physical constants or parameters, or auxiliary or intermediate variables that must relate to the primary variables.

5. If units agree, then your model is dimensionally consistent but may have other shortcomings.

6. The result of step 4 is a differential equation; complex problems can involve extremely complicated models.

7. Make sure your model works.

1.2 Solutions of Some Differential Equations

While slope fields are important, we often need the solution to a differential equation itself. Consider the ODE

$$\frac{dp}{dt} = 0.5p - 450 = \frac{p - 900}{2} \tag{1.3}$$

Rearranging,

$$\frac{dp}{dt} \cdot \frac{1}{p - 900} = \frac{1}{2} \tag{1.4}$$

Recognising that this gives

$$\frac{d}{dt} \ln |p - 900| = \frac{1}{2} \tag{1.5}$$

Integrating and taking exponentials we get

$$p = 900 + ce^{t/2}, \qquad c \in \mathbb{R} \tag{1.6}$$

We get an infinite set of solutions as $c \in \mathbb{R}$; to get a specific solution, we have to

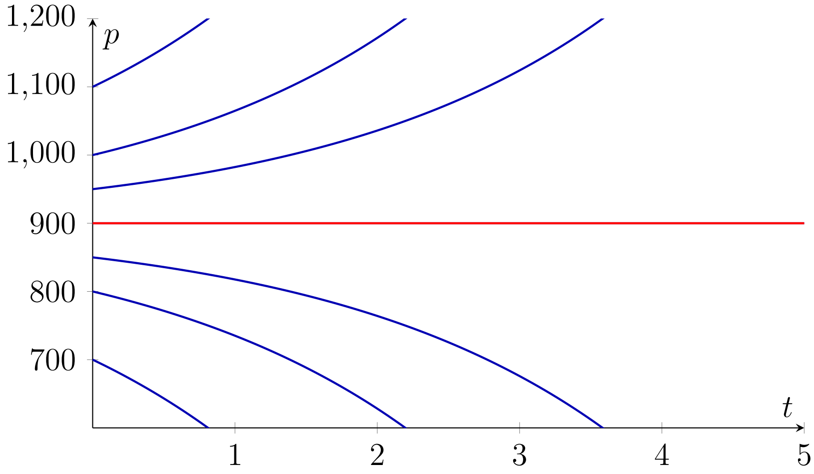


Figure 2: Graphs of $p = 900 + ce^{t/2}$ for several values of c. Each blue curve is a solution of $dp/dt = 0.5p - 450$.

specifically an initial condition like $p(0) = 400$.

Definition 1.3 (Initial Value Problem)**.** An initial value problem (IVP) is a problem where we have a differential equation and an initial solution,

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0 \tag{1.7}$$

The infinite set of solutions is called the *general solution*, and the geometric representation is an infinite family of curves called *integral curves*. Each integral curve is associated with a particular value of c and is a graph of the solution corresponding to that value of c. Specifying an initial condition gives an integral curve that passes through the given initial point.

1.3 Classification of Differential Equations

Definition 1.4 (Ordinary Differential Equation)**.** Ordinary differential equations (ODEs) involve unknown functions that depend on a single independent variable; the derivatives are ordinary,

$$\frac{dy}{dt} = ay - b \tag{1.8}$$

Definition 1.5 (Partial Differential Equation)**.** Partial differential equations (PDEs) involve unknown functions that depend on multiple variables; the derivatives are partial,

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \tag{1.9}$$

If there is only one function we have a single differential equation. If there are multiple functions we have a *system* of differential equations. The dimension of the system equals the number of equations,

$$\frac{dx}{dt} = ax - \alpha xy \tag{1.10}$$

$$\frac{dy}{dt} = -cy + \gamma xy \tag{1.11}$$

Definition 1.6 (Order)**.** The order of a differential equation is the order of the highest derivative that appears in the equation. Writing

$$y^{(n)} = f\left(t, y, y', y'', \ldots, y^{(n-1)}\right) \tag{1.12}$$

gives a differential equation of order n.

Definition 1.7 (Linear Differential Equation)**.** A linear differential equation is of the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t) \tag{1.13}$$

This is commonly reworded as: an ODE is linear if it is of the form

$$F\left(t, y, y', y'', \ldots, y^{(n)}\right) = 0 \tag{1.14}$$

iff F is a linear function of $y, y', \ldots, y^{(n)}$.

An equation not of this form is *nonlinear*. These are much more complicated to solve, but many nonlinear differential equations can be converted into a linear equation in a process called *linearisation*.

Definition 1.8 (Solution)**.** A solution of the nth order differential equation on the interval $\alpha < t < \beta$ is a function ϕ such that $\phi', \phi'', \ldots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(t) = f\left(t, \phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t)\right) \tag{1.15}$$

$\forall t \in (\alpha, \beta)$. We assume $f(t, u(t), \ldots, u^{(n-1)}(t))$ is a real-valued function and $u(t) = \phi$ is a real-valued solution unless otherwise specified.

To verify if a function is a solution to a differential equation, just substitute it into the equation and see if it satisfies it.

We usually ask questions about existence, uniqueness and continuous dependence for initial value problems,

- *Existence:* Does the ODE have a solution?
- *Uniqueness:* If a solution exists, is it unique?
- *Continuous dependence on initial conditions:* If we perturb our initial conditions, what will the solution do? Will the solution change significantly or only slightly?

If we meet all of these, we say our problem is *well-posed*.

2 First-Order Differential Equations

Recall that a first-order DE is of the form

$$\frac{dy}{dt} = f(t, y) \tag{2.1}$$

Any differentiable function $y = \phi(t)$ that satisfies the DE $\forall t$ in some interval is called a *solution*. There is no general method for solving a DE; instead, there are various methods depending on the form of the DE.

2.1 Method of Integrating Factors

Consider the general first-order linear ODE written as

$$\frac{dy}{dt} + p(t)y = g(t) \tag{2.2}$$

Definition 2.1 (Integrating Factor)**.** The method of integrating factors involves multiplying the ODE by a function $\mu(t)$ known as the integrating factor (IF),

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t) \tag{2.3}$$

Notice that the LHS can be written using the product rule

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)\frac{dy}{dt} + y\frac{d\mu}{dt} \tag{2.4}$$

only when

$$\frac{d\mu}{dt} = \mu(t)p(t) \tag{2.5}$$

Rearranging,

$$\frac{1}{\mu(t)}\frac{d\mu}{dt} = p(t) \tag{2.6}$$

This can be rewritten using the chain rule,

$$\frac{d}{dt}\left(\ln|\mu(t)|\right) = p(t) \implies \ln|\mu(t)| = \int p(t)\,dt + c_1 \tag{2.7}$$

Then,

$$\mu(t) = e^{\int p(t)\,dt + c_1} = Ae^{\int p(t)\,dt} \tag{2.8}$$

Without loss of generality we can set $A = 1$ as the constant factors and cancels when we multiply both sides of the ODE by the IF. The IF is then defined as

$$\mu(t) = e^{\int p(t)\,dt} \tag{2.9}$$

Now when we multiply by $\mu(t)$ we have

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)g(t) \tag{2.10}$$

$$\implies \mu(t)y = \int \mu(t)g(t)\,dt + c \tag{2.11}$$

The solution of the ODE is then

$$y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)\,dt + c\right] \tag{2.12}$$

If we have an IVP $y(t_0) = y_0$, we can use it to solve for c,

$$y_0 = \frac{1}{\mu(t_0)}\left[\int \mu(t_0)g(t_0)\,dt + c\right] \tag{2.13}$$

2.2 Variation of Parameters

Variation of parameters is nearly identical to the IF method. Consider the general first-order linear ODE

$$\frac{dy}{dt} + p(t)y = g(t) \tag{2.14}$$

If $g(t) = 0$,

$$\frac{dy}{dt} = -p(t)y \tag{2.15}$$

$$\implies \frac{y'}{y} = -p(t) \tag{2.16}$$

$$\implies y = Ae^{-\int p(t)\,dt} \tag{2.17}$$

Now if $g(t) \neq 0$, we look for a solution of the form

$$y(t) = A(t)e^{-\int p(t)\,dt} \tag{2.18}$$

Plug this into the ODE,

$$\frac{dy}{dt} = A'(t)e^{-\int p(t)\,dt} - p(t)A(t)e^{-\int p(t)\,dt} \tag{2.19}$$

$$\implies A'(t)e^{-\int p(t)\,dt} - p(t)A(t)e^{-\int p(t)\,dt} + p(t)A(t)e^{-\int p(t)\,dt} = g(t) \tag{2.20}$$

Thus,

$$A'(t)e^{-\int p(t)\,dt} = g(t) \tag{2.21}$$

$$\implies A'(t) = g(t)e^{\int p(t)\,dt} \tag{2.22}$$

Recognising that $\mu(t) = e^{\int p(t)\,dt}$, we have

$$A(t) = \int \mu(t)g(t)\,dt + c \tag{2.23}$$

Plug this into our solution,

$$y(t) = e^{-\int p(t)\,dt}\left[\int \mu(t)g(t)\,dt + c\right] \tag{2.24}$$

$$= \frac{1}{\mu(t)}\left[\int \mu(t)g(t)\,dt + c\right] \tag{2.25}$$

Thus, we get the same solution as the method of IF. Both techniques can be used; there is no situation where one technique will work and the other won't.

2.3 Separable Differential Equations

We now consider first-order nonlinear ODEs. These can be written as

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{2.26}$$

Definition 2.2 (Separable Differential Equation)**.** We call a differential equation separable when $M(x, y)$ is only a function of x, $M(x)$, and $N(x, y)$ is only a function of y, $N(y)$,

$$M(x) + N(y)\frac{dy}{dx} = 0 \tag{2.27}$$

Let us define $H_1(x)$ and $H_2(y)$ to be the antiderivatives of M and N respectively. Then,

$$\frac{dH_1}{dx} + \frac{dH_2}{dy}\frac{dy}{dx} = 0 \tag{2.28}$$

Recognising from the chain rule,

$$\frac{dH_2}{dy}\frac{dy}{dx} = \frac{d}{dx}\left[H_2(y(x))\right] \tag{2.29}$$

Then,

$$\frac{d}{dx}\left[H_1(x) + H_2(y(x))\right] = 0 \tag{2.30}$$

which gives

$$H_1(x) + H_2(y(x)) = c \tag{2.31}$$

Any differentiable function of the form $y = \phi(x)$ that satisfies the above is a solution to the nonlinear ODE. If we have an IVP $y(x_0) = y_0$ then

$$H_1(x_0) + H_2(y_0) = c \tag{2.32}$$

This gives

$$
\begin{aligned}
& H_1(x) + H_2(y(x)) = H_1(x_0) + H_2(y_0) && (2.33) \\
\implies & H_1(x) - H_1(x_0) + H_2(y(x)) - H_2(y_0) = 0 && (2.34)
\end{aligned}
$$

From the fundamental theorem of calculus,

$$
\int_{x_0}^{x} M(t)\,dt + \int_{y_0}^{y} N(t)\,dt = 0 \tag{2.35}
$$

This is known as an *implicit solution* of the ODE.

2.4 Homogeneous Differential Equations

Consider the ODE

$$
\frac{dy}{dx} = f(x, y) \tag{2.36}
$$

Definition 2.3 (Homogeneous Differential Equation)**.** If $f(x, y)$ can be written as a function of the ratio y/x, we call the ODE homogeneous,

$$
\frac{dy}{dx} = g\left(\frac{y}{x}\right) \tag{2.37}
$$

We can make the substitution

$$
v = \frac{y}{x} \implies y = xv \implies \frac{dy}{dx} = v + x\frac{dv}{dx} \tag{2.38}
$$

Then,

$$v + x\frac{dv}{dx} = g(v) \tag{2.39}$$

$$\implies x\frac{dv}{dx} = g(v) - v \tag{2.40}$$

$$\implies \frac{1}{g(v)-v}\frac{dv}{dx} = \frac{1}{x} \tag{2.41}$$

$$\implies -\frac{1}{x} + \frac{1}{g(v)-v}\frac{dv}{dx} = 0 \tag{2.42}$$

This is a separable ODE in terms of v; we use the technique in the previous section to solve. We assume $x \neq 0$ and $g(v) - v \neq 0$ to divide. If $g(v) = v$ then we have the trivial case

$$v + x\frac{dv}{dx} = g(v) \implies x\frac{dv}{dx} = 0 \tag{2.43}$$

2.5 Exact Differential Equations

Suppose we have our first-order nonlinear ODE of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{2.44}$$

Definition 2.4 (Exact Differential Equation)**.** Suppose we can identify a function $\psi(x,y)$ such that

$$\frac{\partial\psi}{\partial x} = M(x,y), \qquad \frac{\partial\psi}{\partial y} = N(x,y) \tag{2.45}$$

where $\psi(x,y) = c$ gives us an implicit $y = \phi(x)$ as a differentiable function of x. Then we can write our ODE as

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi(x,\phi(x)) \tag{2.46}$$

Then,

$$\frac{d}{dx}\psi(x,\phi(x)) = 0 \tag{2.47}$$

Solutions to the ODE are given implicitly by $\psi(x, y) = c$.

For complicated equations, it is difficult to tell whether a given equation is exact, and if it is, the method to find $\psi(x, y)$. The following theorem has a constructive proof — the proof gives us a way to find $\psi(x, y)$.

Theorem 2.1 (Exactness)**.** Let the functions M, N, M_y, N_x be continuous in the rectangular region $R : \alpha < x < \beta,\ \gamma < y < \delta$. Then

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{2.48}$$

is an exact ODE if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.49}$$

at each point of R. That is, there exists a function ψ such that

$$\frac{\partial \psi}{\partial x} = M \text{ and } \frac{\partial \psi}{\partial y} = N \tag{2.50}$$

if and only if M and N satisfy $M_y = N_x$.

Proof. Forwards. Assume there is a function ψ with $\psi_x = M$ and $\psi_y = N$. Then

$$M_y = \psi_{xy}, \qquad N_x = \psi_{yx} \tag{2.51}$$

Since M_y and N_x are continuous, then ψ_{xy} and ψ_{yx} are continuous, and they are equal.

Backwards. We need to show that if M and N satisfy $M_y = N_x$ then our ODE is exact, i.e. we need to construct a ψ where $\psi_x = M$ and $\psi_y = N$. If $\partial\psi/\partial x = M(x, y)$ then

$$\psi(x, y) = \int M(x, y)\, dx = Q(x, y) + h(y) \implies \frac{\partial Q}{\partial x} = M \tag{2.52}$$

Then,

$$\frac{\partial \psi}{\partial y} = \frac{\partial Q}{\partial y} + h'(y) = N(x, y) \tag{2.53}$$

Solving for $h'(y)$,

$$h'(y) = N(x, y) - \frac{\partial Q}{\partial y} \tag{2.54}$$

Now $h'(y)$ only depends on y so $N(x, y) - \partial Q/\partial y$ must too. Then,

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial Q}{\partial y}\right] = 0 \tag{2.55}$$

$$\implies \frac{\partial N}{\partial x} - \frac{\partial^2 Q}{\partial x \partial y} = 0 \tag{2.56}$$

$$\implies \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\frac{\partial Q}{\partial x} = 0 \tag{2.57}$$

Since $\partial Q/\partial x = M$, then $\partial N/\partial x - \partial M/\partial y = 0$. Thus, $N_x = M_y$. ■

Example 0

Solve the DE

$$y \cos x + 2xe^y + \left(\sin x + x^2 e^y - 1\right) \frac{dy}{dx} = 0$$

Solution

We identify

$$M(x,y) = y\cos x + 2xe^y, \qquad N(x,y) = \sin x + x^2e^y - 1$$

Check exactness,

$$\frac{\partial M}{\partial y} = \cos x + 2xe^y, \qquad \frac{\partial N}{\partial x} = \cos x + 2xe^y$$

Thus the DE is exact so we can find a $\psi(x,y)$ with $\psi_x = y\cos x + 2xe^y$ and $\psi_y = \sin x + x^2e^y - 1$. Integrating ψ_x with respect to x,

$$\psi = \int (y\cos x + 2xe^y)\ dx = y\sin x + x^2e^y + h(y)$$

Differentiating with respect to y and comparing with ψ_y,

$$\psi_y = \sin x + x^2e^y + h'(y) = \sin x + x^2e^y - 1 \implies h'(y) = -1 \implies h(y) = -y$$

Thus,

$$\psi(x,y) = y\sin x + x^2e^y - y = c$$

gives an implicit solution.

2.6 Exact Differential Equations and Integrating Factors

Remember that for an ODE of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{2.58}$$

it is exact if we can find a ψ where $\psi_x = M$ and $\psi_y = N$. There are cases where $M_y \neq N_x$ so we cannot use the method directly. Thus we make use of an integrating factor $\mu = \mu(x, y)$ and multiply, giving

$$\mu M + \mu N \frac{dy}{dx} = 0 \tag{2.59}$$

Now we need to make it exact, i.e. find a $\psi = \psi(x, y)$ where

$$\frac{\partial \psi}{\partial x} = \mu M, \qquad \frac{\partial \psi}{\partial y} = \mu N \tag{2.60}$$

This occurs when

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \tag{2.61}$$

If this holds we have an exact ODE so we solve as we would with an exact ODE. Now, by the product rule,

$$M \frac{\partial \mu}{\partial y} + \mu \frac{\partial M}{\partial y} = N \frac{\partial \mu}{\partial x} + \mu \frac{\partial N}{\partial x} \tag{2.62}$$

This is a partial differential equation so we assume either μ is only a function of x, $\mu = \mu(x)$. Then,

$$\mu \frac{\partial M}{\partial y} = N \frac{d\mu}{dx} + \mu \frac{\partial N}{\partial x} \tag{2.63}$$

$$\implies N \frac{d\mu}{dx} = \mu \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \tag{2.64}$$

Thus,

$$\frac{\mu'}{\mu} = \frac{M_y - N_x}{N} \tag{2.65}$$

Since μ'/μ is only a function of x, $(M_y - N_x)/N$ depends on x only. If our ODE was originally exact then $M_y - N_x = 0$, so μ is a constant — we are multiplying our exact ODE by a constant, which is still exact.

Now, if we assume μ is a function of y only then $\mu = \mu(y)$ so

$$M \frac{d\mu}{dy} + \mu \frac{\partial M}{\partial y} = \mu \frac{\partial N}{\partial x} \tag{2.66}$$

$$\implies \frac{\mu'}{\mu} = \frac{N_x - M_y}{M} \tag{2.67}$$

μ'/μ depends on y only so $(N_x - M_y)/M$ depends on y only. These are constraints we can place on μ.

Example 1

Solve the ODE via an exact integrating factor

$$e^{2x} + y - 1 - y' = 0$$

Solution

Identify

$$M(x,y) = e^{2x} + y - 1, \qquad N(x,y) = -1$$

Check exactness,

$$M_y = 1, \qquad N_x = 0$$

$M_y \neq N_x$, so the ODE is not exact. Multiply by an IF μ,

$$\mu\left(e^{2x} + y - 1\right) - \mu y' = 0 \implies \mu M + \mu N y' = 0$$

If μ is a function of x,

$$\frac{\mu'}{\mu} = \frac{M_y - N_x}{N} = \frac{1-0}{-1} = -1$$

Thus $\mu'/\mu = -1$, giving

$$\ln|\mu| = \int -1\,dx = -x + c_1 \implies |\mu| = e^{-x}e^{c_1} = Ae^{-x} \implies \mu = e^{-x}$$

Our ODE becomes

$$e^{-x}\left(e^{2x} + y - 1\right) - e^{-x}y' = 0 \implies e^{x} + e^{-x}y - e^{-x} - e^{-x}y' = 0$$

Now identify

$$\psi_x = e^{x} + e^{-x}y - e^{-x}, \qquad \psi_y = -e^{-x}$$

Integrating ψ_y with respect to y,

$$\psi = -e^{-x}y + h(x)$$

Differentiating with respect to x,

$$\psi_x = e^{-x}y + h'(x) = e^{x} + e^{-x}y - e^{-x} \implies h'(x) = e^{x} - e^{-x} \implies h(x) = e^{x} + e^{-x}$$

Thus,

Example 2

Solve the ODE

$$y + (2x - ye^y)\frac{dy}{dx} = 0$$

Solution

Identify

$$M(x, y) = y, \qquad N(x, y) = 2x - ye^y$$

Check exactness,

$$M_y = 1, \qquad N_x = 2$$

$M_y \neq N_x$ so not exact. If μ is a function of x,

$$\frac{\mu'}{\mu} = \frac{M_y - N_x}{N} = \frac{1-2}{2x - ye^y} = \frac{-1}{2x - ye^y}$$

This depends on y, a contradiction, so assume μ is a function of y,

$$\frac{\mu_y}{\mu} = \frac{N_x - M_y}{M} = \frac{2-1}{y} = \frac{1}{y}$$

Then,

$$\ln|\mu| = \int \frac{1}{y}\, dy = \ln|y| + c \implies \mu = y \text{ (WLOG)}$$

Our ODE becomes

$$y^2 + (2xy - y^2 e^y)\frac{dy}{dx} = 0$$

Since this is exact there is a $\psi = \psi(x, y)$ with

$$\frac{\partial \psi}{\partial x} = y^2, \qquad \frac{\partial \psi}{\partial y} = 2xy - y^2 e^y$$

Integrating,

$$\psi = \int y^2\, dx = y^2 x + h(y)$$

Differentiating,

$$\frac{\partial \psi}{\partial y} = 2xy + h'(y) = 2xy - y^2 e^y \implies h'(y) = -y^2 e^y$$

Integrating by parts,

$$h(y) = \int -y^2 e^y\, dy = -y^2 e^y + 2ye^y - 2e^y + c$$

Thus

2.7 Existence and Uniqueness Theorem

In the case of linear ODEs,

$$\frac{dy}{dt} + p(t)y = g(t) \tag{2.68}$$

the uniqueness and existence theorem states:

Theorem 2.2 (Existence and Uniqueness for Linear ODEs)**.** If p and g are continuous on an open interval $I = (\alpha, \beta)$ containing $t = t_0$ where $\alpha < t_0 < \beta$, then there exists a unique function $y = \phi(t)$ that satisfies the ODE for each $t \in I$ with the initial condition $y(t_0) = y_0$ holding for any arbitrarily prescribed $y_0 \in \mathbb{R}$.

Proof. Multiply by the integrating factor $\mu(t) = e^{\int p(t)\,dt}$. Since p is continuous, $p(t)$ is integrable and exists so $\int p(t)\,dt$ is continuously differentiable and non-zero (we don't want to multiply by 0). All of this is $\forall t \in I$. Thus,

$$e^{\int p(t)\,dt}\frac{dy}{dt} + e^{\int p(t)\,dt}p(t)y = e^{\int p(t)\,dt}g(t) \tag{2.69}$$

Then,

$$\frac{d}{dt}\left(ye^{\int p(t)\,dt}\right) = g(t)e^{\int p(t)\,dt} \tag{2.70}$$

$$\therefore ye^{\int p(t)\,dt} = \int g(t)e^{\int p(t)\,dt}\,dt \tag{2.71}$$

$$\implies y = \frac{1}{e^{\int p(t)\,dt}}\left[\int g(t)e^{\int p(t)\,dt}\,dt + c\right] \tag{2.72}$$

Since $e^{\int p(t)\,dt}$ is continuous and $g(t)$ is continuous, the product is continuous and we have a continuous solution. Now that we proved existence we need to prove uniqueness. Consider the IVP,

$$\mu(t) = \mu(t_0)e^{\int_{t_0}^{t} p(s)\,ds} \tag{2.73}$$

WLOG let $\mu(t_0) = 1$ (this will not affect anything as $\mu(t_0)$ can be factored and cancelled). Then,

$$\mu(t) = e^{\int_{t_0}^{t} p(s)\,ds} \tag{2.74}$$

Then,

$$\int_{t_0}^{t} \frac{d}{ds}(\mu y)\, ds = \int_{t_0}^{t} \mu(s)g(s)\, ds \tag{2.75}$$

$$\implies \mu(t)y(t) - \mu(t_0)y(t_0) = \int_{t_0}^{t} \mu(s)g(s)\, ds \tag{2.76}$$

$$\implies y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)\, ds + \mu(t_0)y(t_0)\right] \tag{2.77}$$

This is a unique solution. ■

For general first-order ODEs, the theorem looks like this.

Theorem 2.3 (Existence and Uniqueness for First-Order ODEs)**.** Let f and $\partial f/\partial y$ be continuous in some rectangle $R : \alpha < t < \beta$ and $\gamma < y < \delta$. Assume R contains the point (t_0, y_0), so $\alpha < t_0 < \beta$ and $\gamma < y_0 < \delta$. Then, there is a unique solution $y = \phi(t)$ of the IVP $y' = f(t, y)$ ODE with IC $y(t_0) = y_0$. The solution exists in some interval

$$t_0 - h < t < t_0 + h \tag{2.78}$$

contained in $\alpha < t < \beta$.

Remark. If the ODE is linear then $f(t, y) = -p(t)y + g(t)$ so $\partial f/\partial y = -p(t)$. Then we use the above theorem. The conditions of the second theorem are sufficient but not necessary. The existence but not the uniqueness of solutions can be proved if f is continuous.

A geometrical consequence of the uniqueness of solutions is that the graphs of two solutions cannot intersect each other; otherwise there would be two solutions to the IC.

Vertical asymptotes or other discontinuities in the solution can only occur at points of discontinuity of p or g in the linear case. In the non-linear case it is hard to determine the interval where the solution exists but $y = \phi(t)$ will exist as long as $(t, \phi(t))$ is in $\alpha < t < \beta$ and $\gamma < y < \delta$.

For first-order linear ODEs, we obtain a general solution with an arbitrary constant c and all possible solutions follow by giving a value to c. For non-linear equations, despite having a constant, not all possible solutions may follow.

A consequence of the uniqueness and existence theorem for linear ODEs is that solutions don't exist at points of discontinuity of $g(t)$: since if p and g were continuous on (α, β), we are guaranteed a unique solution.

The case for linear ODEs is a stronger theorem than the general one, since it guarantees a unique solution on any interval where p and g are continuous, while the general theorem only guarantees a unique solution on a small interval $(t_0 - h, t_0 + h)$ around t_0.

2.8 Bernoulli Equations

Definition 2.5 (Bernoulli Equation). A Bernoulli equation is a non-linear ODE of the form

$$\frac{dy}{dt} + p(t)y = g(t)y^n \tag{2.79}$$

If $n = 0$ or $n = 1$ we get a linear equation which can be solved via integrating factors. If $n > 1$, make the substitution $v = y^{1-n}$,

$$\frac{dv}{dt} = (1-n)y^{-n}\frac{dy}{dt} \tag{2.80}$$

Multiply the ODE by y^{-n} to get

$$y^{-n}\frac{dy}{dt} + p(t)y^{1-n} = g(t) \tag{2.81}$$

Then,

$$\frac{1}{1-n}\frac{dv}{dt} + p(t)v = g(t) \tag{2.82}$$

Thus,

$$\frac{dv}{dt} + (1-n)p(t)v = (1-n)g(t) \tag{2.83}$$

This is linear so multiply by the integrating factor

$$\mu(t) = e^{\int (1-n)p(t)\,dt} \tag{2.84}$$

The solution is

$$v(t) = \frac{1}{\mu(t)}\left[\int \mu(t)(1-n)g(t)\,dt + c\right] \tag{2.85}$$

Since $v(t) = y^{1-n}$, we have $y = \sqrt[1-n]{v}$.

2.9 Applications of ODEs

2.9.1 Mixing Problems

Mixing problems are one application of using ODEs to model problems. Assume a fluid flows into a tank at rate r_i (m^3s^{-1}), which contains a substance like salt or dye of concentration (kg m^{-3}), $\gamma(t)$. Assume the fluid is perfectly mixed with the substance and flows out at a rate r_0 (m^3s^{-1}). If $Q(t)$ is the amount of substance in the tank, then the concentration at time t is

$$c(t) = \frac{Q(t)}{V(t)} \quad \text{(concentration of final fluid)} \tag{2.86}$$

where $V(t)$ is the volume of the fluid where

$$\frac{dV}{dt} = r_i - r_0 \implies V(t) = (r_i - r_0)t + V_0 \tag{2.87}$$

where V_0 is the initial volume. Then, the rate of change of the quantity $Q(t)$ is the difference between the amount that flows in and the amount that flows out,

$$\frac{dQ}{dt} = r_i\gamma(t) - r_0\frac{Q(t)}{V(t)} \tag{2.88}$$

If $r_i = r_0$ we get $V(t) = V_0$ to simplify,

$$\frac{dQ}{dt} = r_i\gamma(t) - \frac{r_0}{V_0}Q \tag{2.89}$$

This can be solved via the integrating factor $\mu = e^{\int \frac{r_0}{V_0}\,dt}$.

2.9.2 Compound Interest

Suppose a sum of money S_0 is deposited in a bank that pays an annual interest rate r. The value of the investment at time t, $S(t)$, depends on the frequency with which interest is compounded and the interest rate itself.

The rate of change of value of the investment is equal to the rate at which interest accrues (the growth rate of investment) and is proportional to the current amount,

$$\frac{dS}{dt} = rS, \qquad S(0) = S_0 \tag{2.90}$$

This is a separable ODE which gives

$$S(t) = S_0 e^{rt} \tag{2.91}$$

Now assume if compounding occurs at finite time intervals m times per year. After t years,

$$S(t) = S_0 \left(1 + \frac{r}{m}\right)^{mt} \tag{2.92}$$

where r is the annual interest rate. Now we can see whether, as $m \to \infty$, we get the same result as compounding continuously,

$$\lim_{m\to\infty} S(t) = S_0 \lim_{m\to\infty} \left(1 + \frac{r}{m}\right)^{mt} \tag{2.93}$$

Since $a = e^{\ln(a)}$ we rewrite the above as

$$\lim_{m\to\infty} e^{\ln\left(1+\frac{r}{m}\right)^{mt}} = \lim_{m\to\infty} e^{mt\ln\left(1+\frac{r}{m}\right)} \tag{2.94}$$

By the laws of limits,

$$e^{t\lim_{m\to\infty} m\ln\left(1+\frac{r}{m}\right)} \tag{2.95}$$

This leads to an indeterminate form $(\infty \cdot 0)$ so we have to use L'Hôpital's rule,

$$\lim_{m\to\infty} m \ln\left(1+\frac{r}{m}\right) = \lim_{m\to\infty} \frac{\ln\left(1+\frac{r}{m}\right)}{1/m} \tag{2.96}$$

$$= \lim_{m\to\infty} \frac{\frac{1}{1+r/m}\cdot\frac{-r}{m^2}}{-\frac{1}{m^2}} \tag{2.97}$$

$$= \lim_{m\to\infty} \frac{m^2 r}{m^2+mr} \tag{2.98}$$

$$= \lim_{m\to\infty} \frac{r}{1+r/m} \tag{2.99}$$

$$= r \tag{2.100}$$

Thus,

$$e^{t\lim_{m\to\infty} m\ln\left(1+\frac{r}{m}\right)} = e^{rt} \tag{2.101}$$

Thus the limit converges (as expected) to

$$S(t) = S_0 e^{rt} \tag{2.102}$$

We now consider continuous compounding interest again where we now have deposits or withdrawals that take place at a constant rate k ie

$$\frac{dS}{dt} = rS + k \tag{2.103}$$

We have a deposit when $k > 0$ and a withdrawal for $k < 0$. This is a first order linear ODE

$$\frac{dS}{dt} - rS = k \tag{2.104}$$

The integrating factor is $\mu = e^{\int -r\,dt} = e^{-rt}$. Then,

$$e^{-rt}\frac{dS}{dt} - rSe^{-rt} = ke^{-rt} \tag{2.105}$$

$$\implies \left(Se^{-rt}\right)' = ke^{-rt} \tag{2.106}$$

$$\implies Se^{-rt} = \int ke^{-rt}\,dt + c_1 \tag{2.107}$$

$$Se^{-rt} = -\frac{k}{r}e^{-rt} + c_1 \tag{2.108}$$

$$\implies S = -\frac{k}{r} + c_1e^{rt} \tag{2.109}$$

Substituting $S(0) = S_0$,

$$S(0) = -\frac{k}{r} + c_1 = S_0 \implies c_1 = S_0 + \frac{k}{r} \tag{2.110}$$

Thus the solution to the IVP is

$$S(t) = \left(S_0 + \frac{k}{r}\right)e^{rt} - \frac{k}{r} = S_0e^{rt} + \frac{k}{r}(e^{rt} - 1) \tag{2.111}$$

2.10 Autonomous Differential Equations and Population Dynamics

Definition 2.6 (Autonomous Differential Equation)**.** Autonomous differential equations are those in which the independent variable does not appear explicitly, i.e.

$$\frac{dy}{dt} = f(y) \tag{2.112}$$

Autonomous differential equations are generally used to model the growth or decline of populations. We will see that we can analyse such equations geometrically and obtain important qualitative information without solving them.

2.10.1 Exponential Growth

Exponential growth is modelled by

$$\frac{dy}{dt} = ry \implies y = ce^{rt} \tag{2.113}$$

where $y = \phi(t)$ is the population of a species at time t and r is the rate of growth or decline.

- If $r > 0$ we have population growth.
- If $r < 0$ we have population decline.

If $y(0) = y_0$ we have $y(t) = y_0 e^{rt}$. This is an unrealistic model as it assumes that the population of a species grows indefinitely when in fact other factors like availability of food, space, etc., eventually show up and reduce the rate of growth. To account for this, logistic growth was proposed.

2.10.2 Logistic Growth

Logistic growth accounts for the fact that the growth rate depends on population given by the logistic equation

$$\frac{dy}{dt} = h(y)y \tag{2.114}$$

Initially, we can ignore resources on populations so for small y, $h(y) \approx r > 0$. $h(y)$ decreases as y increases to show decreasing growth. When y is sufficiently large, $h(y) < 0$ to show population decreases due to resource limitations. The simplest choice for $h(y)$ that satisfies these is

$$h(y) = r - ay \tag{2.115}$$

Thus

$$\frac{dy}{dt} = (r - ay)y = r\left(1 - \frac{y}{K}\right)y, \qquad K = \frac{r}{a} \tag{2.116}$$

$K = r/a > 0$ is known as the saturation level or environmental carrying capacity, and r is known as the intrinsic growth rate (the growth rate in the absence of other factors).

The equilibrium solutions are solutions that are constant in time (solution doesn't vary in t). These occur at $dy/dt = 0$, which gives

$$\phi_1(t) = 0 \text{ or } \phi_2(t) = K \tag{2.117}$$

We can plot $f(y) = r(1 - y/K)y$ against y to get

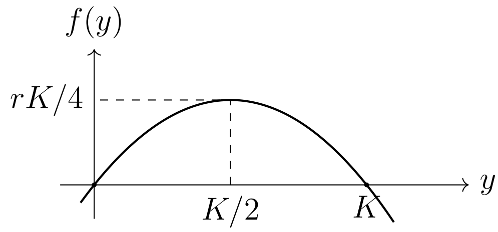


Figure 3: Graph of $f(y) = r(1 - y/K)y$ for the logistic equation.

We see that $dy/dt > 0$ for $0 < y < K$ and $dy/dt < 0$ for $y > K$. Thus solutions with initial conditions in $(0, K)$ increase until they reach the line $y = K$ (because $dy/dt > 0$), while solutions with initial conditions in (K, ∞) decrease until they reach $y = K$ (because $dy/dt < 0$).

We can consider the concavity of y by considering d^2y/dt^2,

$$\frac{d^2y}{dt^2} = \frac{d}{dt} f(y) \tag{2.118}$$

$$= \frac{df}{dy}\frac{dy}{dt} \tag{2.119}$$

$$= \frac{df}{dy} f(y) \tag{2.120}$$

For $f(y) = ry - ry^2/K$, $df/dy = r - 2ry/K$, so

$$\frac{d^2y}{dt^2} = r\left(1 - \frac{2y}{K}\right)\left(1 - \frac{y}{K}\right)y \tag{2.121}$$

The curve is concave up ($d^2y/dt^2 > 0$) for $(0, K/2)$ and concave down in $(K/2, K)$, and concave up in (K, ∞). We can summarise it in

y	$0 < y < K/2$	$K/2 < y < K$	$y > K$
$y' = f$	$+$	$+$	$-$
df/dy	$+$	$-$	$-$
y''	$+$	$-$	$+$

This can be used to draw solution curves.

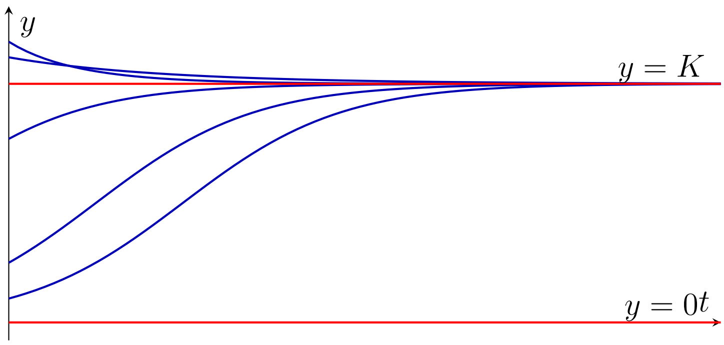


Figure 4: Solution curves to the logistic equation.

These solutions do not intersect $y = K$ as the existence and uniqueness theorem states only one solution passes a given point. Also a solution that starts out in $0 < y < K$ remains in this interval for all t, and so does a solution that starts out at $K < y < \infty$.

The logistic equation is separable so we can solve it,

$$y' = r\left(1 - \frac{y}{K}\right)y = ry - \frac{ry^2}{K} = \frac{rKy - ry^2}{K} = ry\left(\frac{K-y}{K}\right) \tag{2.122}$$

$$\implies \frac{y'}{y(K-y)} = \frac{r}{K} \tag{2.123}$$

$$\implies \int \frac{1}{y(K-y)}\,dy = \int \frac{r}{K}\,dt = \frac{r}{K}t + c_1 \tag{2.124}$$

By partial fractions,

$$\frac{1}{y(K-y)} = \frac{A}{y} + \frac{B}{K-y} = \frac{1}{K}\left(\frac{1}{y} + \frac{1}{K-y}\right) \tag{2.125}$$

Then,

$$\int \frac{1}{K}\left(\frac{1}{y}+\frac{1}{K-y}\right) dy = \frac{r}{K}t + c_1 \tag{2.126}$$

$$\implies \frac{1}{K}\left(\ln|y| - \ln|K-y|\right) = \frac{r}{K}t + c_1 \tag{2.127}$$

$$\implies \ln\left|\frac{y}{K-y}\right| = rt + c_2 \tag{2.128}$$

$$\implies \frac{y}{K-y} = Ce^{rt} \tag{2.129}$$

$$\implies y = CKe^{rt} - Cye^{rt} \tag{2.130}$$

$$\implies y = \frac{CKe^{rt}}{1+Ce^{rt}} \tag{2.131}$$

At $y(0) = y_0$ we have

$$y_0 = \frac{CK}{1+C} \implies C = \frac{y_0}{K-y_0} \tag{2.132}$$

Thus,

$$y(t) = \frac{y_0Ke^{rt}}{K-y_0+y_0e^{rt}} = \frac{y_0K}{y_0+(K-y_0)e^{-rt}} \tag{2.133}$$

2.10.3 Phase Lines and Stability

A *phase line* is a way to visualise solutions to autonomous ODEs.

1. For $dy/dt = f(y)$, plot $f(y)$ against y.
2. Mark all the equilibrium solutions on a vertical line.
3. Draw an arrow up (if $f(y) > 0$) or an arrow down (if $f(y) < 0$) depending on the sign of $f(y)$ in these regions [plug equilibrium solution in $f(y)$ to get sign].

The phase line for the logistic equation is as follows.

We can classify the stability of solutions from the phase diagram; the stability describes how solutions behave under small perturbations.

Definition 2.7 (Stable Equilibrium)**.** A stable equilibrium solution says that the solution tends to stay close to the equilibrium for all future times regardless of small

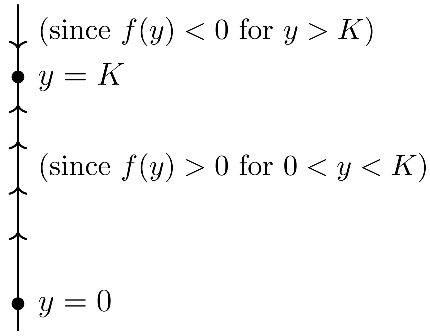


Figure 5: Phase line for the logistic equation.

perturbations.

Definition 2.8 (Asymptotically Stable Equilibrium)**.** A stronger condition is an asymptotically stable equilibrium which says the equilibrium is stable and nearby solutions approach the equilibrium point as $t \to \infty$.

Definition 2.9 (Unstable Equilibrium)**.** An unstable equilibrium is when small perturbations cause solutions to move away from the equilibrium solution.

Definition 2.10 (Semistable Equilibrium)**.** A semistable equilibrium is an equilibrium stable on one side but unstable on another.

2.11 Proof of the Existence and Uniqueness Theorem

This is a proof of the existence and uniqueness theorem for any first order ODE (both linear and non-linear). The theorem is also known as the Picard–Lindelöf theorem.

Theorem 2.4 (Picard–Lindelöf)**.** For the IVP $y' = f(t, y),\ y(t_0) = y_0$, if f and $\partial f/\partial y$ are continuous on a rectangle $R : \alpha < t < \beta,\ \gamma < y < \delta$ that contains the initial condition (t_0, y_0), then there exists a unique function $\phi(t)$ that solves the IVP and exists for $t_0 - h < t < t_0 + h$.

Proof. **Step 1.** It is sufficient to prove the theorem for the IVP $y' = f(t, y),\ y(0) = 0$ because we can always shift to the more general form $y(t_0) = y_0$ with the following transformations. Define

$$z(t) = y(t + t_0) - y_0 \tag{2.134}$$

$$g(t, z) = f(t + t_0,\ z + y_0) \tag{2.135}$$

Then,

$$z'(t) = y'(t + t_0) \tag{2.136}$$

$$= f(t + t_0,\ y(t + t_0)) \tag{2.137}$$

$$= f(t + t_0,\ z(t) + y_0) \tag{2.138}$$

$$= g(t, z(t)) \tag{2.139}$$

When $t = 0$, $z(0) = y(t_0) - y_0 = 0$. So the new function satisfies $z'(t) = g(t, z)$ for $z(0) = 0$. Thus, if we prove the theorem for the IVP $y(0) = 0$, we can always just make a shift like above to get to any initial condition.

Step 2. The current problem is not very helpful to work with; thus we want to convert our IVP into something we can work with. Assume there is a function $y = \phi(t)$ that satisfies the IVP, i.e. $\phi' = f(t, \phi)$ and $\phi(0) = 0$. Thus we can integrate to get ϕ,

$$\phi(t) = \int_0^t f(s, \phi(s))\, ds = \phi(t) - \phi(0) \tag{2.140}$$

This is called an *integral equation*. We used the fact that since ϕ solves the IVP, $\phi(0) = 0$. Additionally now suppose there is a continuous function that satisfies the integral equation. Since $\phi(t) = \int_0^t f(s, \phi(s))\, ds$ then $\phi(0) = \int_0^0 f(s, \phi(s))\, ds = 0$. Then $\phi(0) = 0$ and via the fundamental theorem of calculus

$$\phi'(t) = f(t, \phi(t)) \tag{2.141}$$

Thus, the IVP and integral equation are equivalent; if we find a unique solution to the integral equation, we have proved the theorem for the IVP.

Step 3. We now try and build a solution via repeated integration. We show that all the solutions then converge to the unique solution ϕ. This is known as *Picard's integration method*. We are trying to build a sequence of functions (each function need not converge to the actual solution, but the entire function will). We start by defining

$$\phi_0(t) = 0 \tag{2.142}$$

This satisfies the initial condition but not necessarily the ODE. Now define

$$\phi_1(t) = \int_0^t f(s, \phi_0(s))\, ds \tag{2.143}$$

$$\phi_2(t) = \int_0^t f(s, \phi_1(s))\, ds \tag{2.144}$$

In general,

$$\phi_n(t) = \int_0^t f(s, \phi_{n-1}(s))\, ds \tag{2.145}$$

This yields us a sequence of functions $\{\phi_n\}$. Each member does not satisfy the ODE but it does satisfy the IC $\phi_n(0) = 0$. The reason it satisfies the IC is because $\phi_n(0) = \int_0^0 f(s, \phi_{n-1}(s))\, ds = 0$. The reason it doesn't satisfy the ODE is because

$$\phi_n'(t) = f(t, \phi_{n-1}(t)) \text{ not } f(t, \phi_n(t)) \tag{2.146}$$

However, if for some k it holds that $\phi_{k+1}(t) = \phi_k(t)$, then $\phi_k(t)$ is a solution. This is because

$$\phi_{k+1}(t) = \int_0^t f(s, \phi_k(s))\, ds = \phi_k(t) \tag{2.147}$$

Thus $\phi_k(t)$ satisfies the integral equation. Also, $\phi_k'(t) = f(t, \phi_k(t))$ and $\phi_k(0) = 0$ so it satisfies the IVP and is thus a true solution. However, in general this does not occur, i.e. we do not get a $\phi_k(t)$ after some finite iteration. They only approach equality as $n \to \infty$ and thus we need to take the limit,

$$\phi(t) = \lim_{n\to\infty} \phi_n(t) \tag{2.148}$$

If the limit exists, $\phi(t)$ will satisfy the integral equation. In essence, the iteration converges to the true solution.

Step 4. We don't know yet if all the integrals from the iteration lie in the domain where f is defined. For example, for some $n = k$ it is possible for the graph of $y = \phi_k(t)$ to lie outside the rectangle R. Then, computing $\phi_{k+1}(t)$ would be impossible since $f(t, y)$ is not necessarily continuous outside R.

To avoid this, restrict t to an even smaller interval than $|t| \leq a$. To find such an interval, we make use of the extreme value theorem, which says a continuous function on a closed and bounded region is bounded by a maximum value M, i.e.

$$|f(t, y)| \leq M \quad \forall (t, y) \in R \tag{2.149}$$

Thus M is a uniform bound on how large f can get inside R. We can use the bound to control how large each iteration can get.

$$|\phi_{n+1}(t)| = \left| \int_0^t f(s, \phi_n(s))\, ds \right| \leq \int_0^{|t|} |f(s, \phi_n(s))|\, ds \tag{2.150}$$

This is true since $|\int f(x)\, dx| \leq \int |f(x)|\, dx$. Also, since $|f(s, \phi_n(s))| \leq M$, then

$$|\phi_{n+1}(t)| \leq \int_0^{|t|} M\, ds = M|t| \tag{2.151}$$

Thus, for each n,

$$|\phi_{n+1}(t)| \leq M|t| \tag{2.152}$$

Now we need to guarantee that the graph of $\phi_{n+1}(t)$ always lies inside R, i.e. $|t| \leq a$ (stays within horizontal bounds of R) and $|\phi_{n+1}(t)| \leq b$ (stays within vertical bounds of R). From the inequality above this holds provided

$$M|t| \leq b \implies |t| \leq \frac{b}{M} \quad (\text{so } |t| \leq a \text{ and } |t| \leq \frac{b}{M}) \tag{2.153}$$

Thus if we define h as

$$h = \min\left(a, \frac{b}{M}\right) \tag{2.154}$$

Then for all t such that $|t| \leq h$, all the points in the graph of $\phi_{n+1}(t)$ lie in R. This ensures that the argument $(s, \phi_n(s))$ in $f(s, \phi_n(s))$ stays in the region where f is known to be continuous and bounded.

Step 5. Now that each member of the sequence $\{\phi_n(t)\}$ is bounded, we need to make sure the sequence converges. Identify

$$\phi_n(t) = \phi_1(t) + (\phi_2(t) - \phi_1(t)) + \cdots + (\phi_n(t) - \phi_{n-1}(t)) \tag{2.155}$$

This is the nth partial sum of the infinite series

$$\phi_1(t) + \sum_{k=1}^{\infty}(\phi_{k+1}(t) - \phi_k(t)) \tag{2.156}$$

$\{\phi_n\}$ converges only if the infinite series converges. Thus, it is necessary to estimate the magnitude of the general term $\phi_{k+1}(t) - \phi_k(t)$.

Corollary 2.4.1 (Lipschitz Condition)**.** If $\partial f/\partial y$ is continuous on a rectangle D, then there is a positive constant K such that

$$|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2| \tag{2.157}$$

where (t, y_1) and $(t, y_2) \in D$. This inequality is known as the Lipschitz condition and K is known as the Lipschitz constant.

Proof. Hold t fixed; consider $F(y) = f(t, y)$. By assumption $F'(y)$ exists and is continuous $\forall y$ such that $(t, y) \in D$. Then by the mean value theorem, for any y_1, y_2 such that (t, y_1) and $(t, y_2) \in D$, $\exists y^* \in (y_1, y_2)$ such that

$$F(y_2) - F(y_1) = F'(y^*)(y_2 - y_1) \tag{2.158}$$

Since F' is continuous on $D_y = \{y \mid (t, y) \in D\}$ then there is a positive constant K such that

$$K = \max_{(t,y)\in D} \left|\frac{\partial f}{\partial y}\right| = \max_{(t,y)\in D} |F'| \tag{2.159}$$

Thus,

$$|F(y_2) - F(y_1)| \leq K|y_2 - y_1| \iff |f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2| \tag{2.160}$$

■

The Lipschitz condition says that no matter where you are in a region, changing y by a little changes f at most K times that amount. Thus, f doesn't oscillate or blow up quickly with respect to y.

Corollary 2.4.2. If $\phi_{n-1}(t)$ and $\phi_n(t)$ belong to the sequence $\{\phi_n(t)\}$, then by the Lipschitz condition and the rectangle defined as above, it follows that

$$|f(t, \phi_n(t)) - f(t, \phi_{n-1}(t))| \leq K|\phi_n(t) - \phi_{n-1}(t)| \tag{2.161}$$

If $|t| \leq h$ and $t \geq 0$ (argument symmetric for $t < 0$),

$$|\phi_1(t) - \phi_0(t)| = |\phi_1(t) - 0| = |\phi_1(t)| \tag{2.162}$$

$$|\phi_1(t)| = \left|\int_0^t f(s, \phi_0(s))\, ds\right| \tag{2.163}$$

$$= \left|\int_0^t f(s, 0)\, ds\right| \tag{2.164}$$

$$\leq \int_0^t M\, ds \leq Mt \tag{2.165}$$

Since $|f(t,y)| \leq M$ for $(t,y) \in D$. Now,

$$\begin{aligned}
|\phi_2(t) - \phi_1(t)| &= \left| \int_0^t f(s, \phi_1(s))\, ds - \int_0^t f(s, \phi_0(s))\, ds \right| &(2.166)\\
&= \left| \int_0^t f(s, \phi_1(s)) - f(s, \phi_0(s))\, ds \right| &(2.167)\\
&\leq \int_0^t |f(s, \phi_1(s)) - f(s, \phi_0(s))|\, ds &(2.168)\\
&\leq \int_0^t K|\phi_1(s) - \phi_0(s)|\, ds &(2.169)
\end{aligned}$$

Since $|\phi_1(t)| \leq Mt$, we have

$$|\phi_2(t) - \phi_1(t)| \leq K \int_0^t |\phi_1(s)|\, ds \leq K \int_0^t Ms\, ds = \frac{KMt^2}{2} \tag{2.170}$$

Thus we see

$$\begin{aligned}
|\phi_1(t) - \phi_0(t)| &\leq Mt &(2.171)\\
|\phi_2(t) - \phi_1(t)| &\leq \frac{KMt^2}{2} &(2.172)
\end{aligned}$$

In general,

$$|\phi_n(t) - \phi_{n-1}(t)| \leq \frac{MK^{n-1}t^n}{n!} \tag{2.173}$$

Since we know $\forall t,\ |t| \leq h$, we can further bound this by

$$|\phi_n(t) - \phi_{n-1}(t)| \leq \frac{MK^{n-1}h^n}{n!} \tag{2.174}$$

We can prove this using mathematical induction (we showed it is true for the base case $n = 1$ and 2). For our inductive hypothesis, we assume it is true for some $n = m$, i.e.

$$|\phi_m(t) - \phi_{m-1}(t)| \leq \frac{MK^{m-1}t^m}{m!} \tag{2.175}$$

Now we show it is true for $n = m + 1$,

$$
\begin{aligned}
|\phi_{m+1}(t) - \phi_m(t)| &= \left| \int_0^t f(s, \phi_m(s))\, ds - \int_0^t f(s, \phi_{m-1}(s))\, ds \right| && (2.176)\\
&= \left| \int_0^t f(s, \phi_m(s)) - f(s, \phi_{m-1}(s))\, ds \right| && (2.177)\\
&\le \int_0^t |f(s, \phi_m(s)) - f(s, \phi_{m-1}(s))|\, ds && (2.178)
\end{aligned}
$$

Add a Lipschitz condition for f, i.e. $|f(t, y_1) - f(t, y_2)| \le K|y_1 - y_2|$,

$$
\begin{aligned}
|\phi_{m+1}(t) - \phi_m(t)| &\le \int_0^t K|\phi_m(s) - \phi_{m-1}(s)|\, ds && (2.179)\\
&\le \int_0^t \frac{K \cdot MK^{m-1}s^m}{m!}\, ds \quad \text{(apply the inductive hypothesis)} && (2.180)\\
&= \frac{MK^m}{m!} \int_0^t s^m\, ds && (2.181)\\
&= \frac{MK^m t^{m+1}}{(m+1)!} && (2.182)\\
&\le \frac{MK^m h^{m+1}}{(m+1)!} && (2.183)
\end{aligned}
$$

Hence proved by mathematical induction.

Now we need to prove the convergence of $\{\phi_n(t)\}$. Write

$$\phi_n(t) = \phi_1(t) + (\phi_2(t) - \phi_1(t)) + \cdots + (\phi_n(t) - \phi_{n-1}(t)) \tag{2.184}$$

Use $|a + b| \le |a| + |b|$,

$$|a + b + c| \le |a| + |b + c| \le |a| + |b| + |c| \tag{2.185}$$

$$|\phi_n(t)| \le |\phi_1(t)| + |\phi_2(t) - \phi_1(t)| + \cdots + |\phi_n(t) - \phi_{n-1}(t)| \tag{2.186}$$

Now,

$$|\phi_n(t)| \le Mt + \frac{MKt^2}{2!} + \cdots + \frac{MK^{n-1}t^n}{n!} \tag{2.187}$$
$$\le \frac{M}{K}\left(Kt + \frac{(Kt)^2}{2!} + \cdots + \frac{(Kt)^n}{n!}\right) \tag{2.188}$$
$$\le \frac{M}{K}\sum_{j=1}^{n}\frac{(Kt)^j}{j!} \tag{2.189}$$
$$\le \frac{M}{K}\sum_{j=0}^{n}\frac{(Kt)^j}{j!} \tag{2.190}$$

Remember $e^x = \sum_{j=0}^{\infty} x^j/j!$ (Taylor series for e^x); this has a radius of convergence of ∞, i.e. $(-\infty, \infty)$.

ϕ_n being a partial sum of a convergent series, converges! This converges absolutely because of absolute value. Absolute convergence implies convergence so ϕ_n converges. The limiting function is

$$\phi(t) = \lim_{n\to\infty} \phi_n(t) \tag{2.191}$$

Step 6. We have now shown there exists a limit function $\phi(t)$. We now need to show $\phi(t)$ satisfies the ODE by showing that it satisfies the integral equation. We would like to know that $\phi(t)$ is continuous. It turns out this is true since $\phi_n \to \phi$ uniformly.

Definition 2.11 (Uniform Convergence)**.** A sequence of functions $\{f_n\}$ is uniformly convergent to a function F in D if and only if

$$\lim_{n\to\infty} |F(x) - f_n(x)| = 0 \quad \forall x = (t, y) \in D \tag{2.192}$$

We proved the uniform convergence of ϕ_n to ϕ in step 5. Now we have the following theorem for calculus:

Theorem 2.5 (Continuity of Uniform Limit)**.** If $\{f_n\}$ converges to F uniformly on D and each function f_n is continuous on D, then F is continuous on D.

Proof. We prove F is continuous at an arbitrary point $x_0 \in D$. Given $\epsilon > 0$, choose N so that

$$|F(x) - f_N(x)| < \frac{\epsilon}{3} \quad \forall x \in D \tag{2.193}$$

Then, for any $x_1 \in D$, it holds

$$|F(x_1) - F(x_0)| = |F(x_1) - f_N(x_1) + f_N(x_1) - f_N(x_0) + f_N(x_0) - F(x_0)| \tag{2.194}$$

$$\leq |F(x_1) - f_N(x_1)| + |f_N(x_1) - f_N(x_0)| + |f_N(x_0) - F(x_0)| \tag{2.195}$$

$$< \frac{\epsilon}{3} + |f_N(x_1) - f_N(x_0)| + \frac{\epsilon}{3} \tag{2.196}$$

Now we use the fact that f_N is continuous on D. Given $\epsilon > 0$, we may choose $\delta > 0$ so that for $|x_0 - x_1| < \delta$ it holds

$$|f_N(x_1) - f_N(x_0)| < \frac{\epsilon}{3} \tag{2.197}$$

Hence,

$$|F(x_1) - F(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \tag{2.198}$$

We have therefore shown that given $\epsilon > 0$, there is a $\delta > 0$ such that for $|x_0 - x_1| < \delta$ it holds

$$|F(x_1) - F(x_0)| < \epsilon \tag{2.199}$$

which proves F is continuous at x_0. ■

Returning to our integral equation,

$$\phi_n(t) = \int_0^t f(s, \phi_{n-1}(s))\, ds \tag{2.200}$$

Take the limit as $n \to \infty$ on both sides to get

$$\phi(t) = \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \int_0^t f(s, \phi_{n-1}(s))\, ds \tag{2.201}$$

The following theorem states we can interchange the operation of integration and taking the limit.

Theorem 2.6 (Integration and Uniform Limits)**.** Let $I = [t_1, t_2]$ be a closed bounded interval on the real line which has nonzero length, and let $\{f_n\}$ be continuous on I. Then, if $f_n \to F$ uniformly on I, it holds

$$\lim_{n\to\infty} \int_{t_1}^{t_2} f_n(s)\, ds = \int_{t_1}^{t_2} \lim_{n\to\infty} f_n(s)\, ds = \int_{t_1}^{t_2} F(s)\, ds \tag{2.202}$$

Proof. Since f_n and F are continuous on I, the integrals exist. For any n we have

$$\left| \int_{t_1}^{t_2} F(s)\, ds - \int_{t_1}^{t_2} f_n(s)\, ds \right| = \left| \int_{t_1}^{t_2} F(s) - f_n(s)\, ds \right| \tag{2.203}$$

$$\leq \int_{t_1}^{t_2} |F(s) - f_n(s)|\, ds \tag{2.204}$$

$$\leq \max_{t\in I} |F(t) - f_n(t)| \times \text{length}(I) \tag{2.205}$$

This is true since $|F(s) - f_n(s)| \leq M$ by the extreme value theorem on $[t_1, t_2]$ so

$$\int_{t_1}^{t_2} |F(s) - f_n(s)|\, ds \leq \int_{t_1}^{t_2} M\, ds = M(t_2 - t_1) \tag{2.206}$$

Since $f_n \to F$ uniformly on I, $\max_{t\in I} |F(t) - f_n(t)| \to 0$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \int_{t_1}^{t_2} f_n(s)\, ds = \int_{t_1}^{t_2} F(s)\, ds \tag{2.207}$$

■

Thus we can show

$$\phi(t) = \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \int_0^t f(s, \phi_{n-1}(s))\, ds \tag{2.208}$$

$$= \int_0^t \lim_{n\to\infty} f(s, \phi_{n-1}(s))\, ds \tag{2.209}$$

$$= \int_0^t f(s, \phi(s))\, ds \tag{2.210}$$

Thus $\phi(t)$ satisfies the integral equation and thus the IVP. We have shown there exists a solution to the IVP.

Step 7. We now need to show $\phi(t)$ is unique, i.e. there is no other solution. Assume a second solution $y = \psi(t)$ holds. Then,

$$\phi(t) - \psi(t) = \int_0^t f(s, \phi(s)) - f(s, \psi(s))\, ds \tag{2.211}$$

$$\implies |\phi(t) - \psi(t)| \leq \int_0^t |f(s, \phi(s)) - f(s, \psi(s))|\, ds \tag{2.212}$$

$$\implies |\phi(t) - \psi(t)| \leq K \int_0^t |\phi(s) - \psi(s)|\, ds \text{ for } 0 \leq t \leq h \tag{2.213}$$

Define U as

$$U(t) = \int_0^t |\phi(s) - \psi(s)|\, ds \tag{2.214}$$

Then $U(0) = \int_0^0 |\phi(s) - \psi(s)|\, ds = 0$. Also, $U(t) \geq 0$ for $t \geq 0$. U is also differentiable by the fundamental theorem of calculus since ϕ and ψ are continuous. Thus,

$$U'(t) = |\phi(t) - \psi(t)| \tag{2.215}$$

Thus, it follows

$$|\phi(t) - \psi(t)| \leq K \int_0^t |\phi(s) - \psi(s)|\, ds \tag{2.216}$$

$$\implies U'(t) - KU(t) \leq 0 \tag{2.217}$$

This is a linear first order ODE in U so multiply by the integrating factor $e^{\int -K\, dt} = e^{-Kt}$,

$$\left(e^{-Kt}U(t)\right)' \leq 0 \tag{2.218}$$

$$\implies e^{-Kt}U(t) \leq 0 \tag{2.219}$$

$$\implies U(t) \leq 0 \tag{2.220}$$

But from the definition of $U(t)$, $U(t) \geq 0$ for each $t \geq 0$. Thus, $U(t) = 0 \implies U'(t) = 0$. Thus,

$$|\phi(t) - \psi(t)| = 0 \implies \phi(t) = \psi(t) \tag{2.221}$$

This shows that there cannot be two different solutions of the IVP for $t \geq 0$. A similar argument can be used for $t < 0$. Thus, we have proved that $\phi(t)$ exists and is unique. This concludes the existence and uniqueness theorem for first order ODEs. ■

2.12 Euler's Method

Only a very small number of ODEs can be solved analytically. Most ODEs, despite the existence of a unique solution, cannot be solved analytically. We can instead use numerical methods to approximate the solution to an IVP. The simplest numerical method is Euler's Method. Consider the IVP

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{2.222}$$

We know the solution passes through (t_0, y_0) and the slope of the tangent line of the solution is $f(t_0, y_0)$. Thus, the equation of the tangent line at (t_0, y_0) is

$$y - y_0 = f(t_0, y_0)(t - t_0) \tag{2.223}$$

If t_1 is close to t_0, we can approximate $y_1 \approx y(t_1)$ by

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0) \tag{2.224}$$

If t_2 is close to t_1, we can approximate $y(t_2)$ by

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1) \tag{2.225}$$

In general we can approximate $y(t_{n+1})$ by

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n) \tag{2.226}$$

If $t_0, t_1, t_2, \ldots, t_n, t_{n+1}$ are uniformly apart we define the *step-size* as

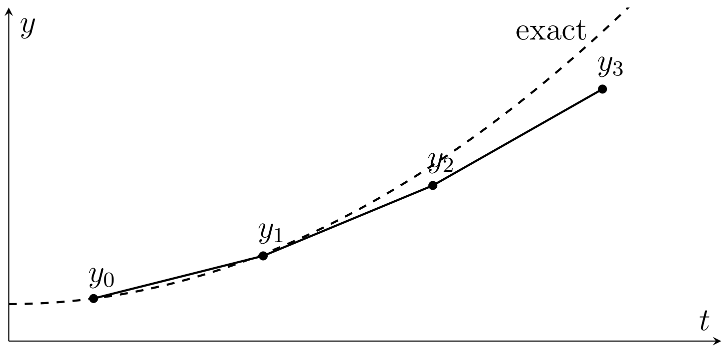


Figure 6: Euler's method: successive tangent segments approximate the true solution curve.

$$h = t_{n+1} - t_n \tag{2.227}$$

Then,

$$y_{n+1} = y_n + hf(t_n, y_n) \tag{2.228}$$

In the interval $[t_n, t_{n+1}]$ the solution is approximated by

$$y(t) = y_n + f(t_n, y_n)(t - t_n) \tag{2.229}$$

As the step-size $h \to 0$, we better approximate the solution.

Example 3

For $y' = y(3 - ty),\ y(0) = 1/2$ we have the following table of values.

	t_n	y_n	$f_n = y_n(3 - t_n y_n)$
$n = 0$	0	0.5	1.5
$n = 1$	0.25	0.875	2.434
$n = 2$	0.5	1.483	3.350
$n = 3$	0.75	2.321	2.923
$n = 4$	1	3.052	-0.1574
$n = 5$	1.25	3.0122	-2.305
$n = 6$	1.5	2.436	

3 Second-Order ODEs

3.1 Homogeneous Differential Equations with Constant Coefficients

A second order ODE is of the form

$$f(t, y, y', y'') = 0 \tag{3.1}$$

A linear second order ODE is of the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \tag{3.2}$$

For a second order ODE, we need two initial conditions (because you end up with two constants of integration). In general for an nth order ODE you need n initial conditions. For a second order ODE, the initial conditions tend to be of the form

$$y(t_0) = y_0 \quad \text{(point the solution curve passes through)} \tag{3.3}$$

$$y'(t_0) = y'_0 \quad \text{(slope of the solution curve at that point)} \tag{3.4}$$

or

$$y(t_0) = y_0 \tag{3.5}$$

$$y(t_1) = y_1 \tag{3.6}$$

A homogeneous second order ODE is of the form

$$y'' + p(t)y' + q(t)y = 0 \tag{3.7}$$

A homogeneous second order ODE has constant coefficients if our ODE is

$$ay'' + by' + cy = 0, \qquad a, b, c \in \mathbb{R} \tag{3.8}$$

A certain set of functions tend to repeat their forms after multiple higher order derivatives — those are known as elementary functions. They tend to be solutions of n-order ODEs for $n > 1$. The elementary functions are:

- Trigonometric: $\sin(at)$, $\cos(at)$
- Exponential: e^{at}
- Polynomials

Assume $y = e^{\lambda t}$ is a solution to the constant coefficient homogeneous ODE, i.e.

$$a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} = 0 \tag{3.9}$$

$$\implies e^{\lambda t}(a\lambda^2 + b\lambda + c) = 0 \tag{3.10}$$

Since $e^{\lambda t} > 0$, we have

$$a\lambda^2 + b\lambda + c = 0 \tag{3.11}$$

We can now solve this quadratic to find values of λ that give us a solution to the ODE. The polynomial equation we obtained is called the *characteristic polynomial* of the ODE. If our characteristic polynomial has two real roots λ_1 and λ_2 then

$$y_1 = e^{\lambda_1 t}, \qquad y_2 = e^{\lambda_2 t} \tag{3.12}$$

are solutions to the ODE. Any linear combination of y_1 and y_2 is also a solution.

Lemma 3.1. If y_1 and y_2 are solutions to a second order homogeneous ODE with constant coefficients $ay'' + by' + cy = 0$, any linear combination of y_1 and y_2 is also a solution.

Proof. Let $y = c_1 y_1 + c_2 y_2$ be a solution of the ODE. Then

$$y'' = c_1 y_1'' + c_2 y_2'', \qquad y' = c_1 y_1' + c_2 y_2' \tag{3.13}$$

$$\begin{aligned} ay'' + by' + cy &= a[c_1 y_1'' + c_2 y_2''] + b[c_1 y_1' + c_2 y_2'] + c[c_1 y_1 + c_2 y_2] && (3.14)\\ &= c_1[ay_1'' + by_1' + cy_1] + c_2[ay_2'' + by_2' + cy_2] && (3.15)\\ &= 0 \quad \text{(since } y_1, y_2 \text{ are solutions)} && (3.16) \end{aligned}$$

Thus $y = c_1 y_1 + c_2 y_2$ is also a solution. ■

The general solution to our ODE is then

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \tag{3.17}$$

3.2 Solutions of Linear Homogeneous Equations

Definition 3.1 (Linear Differential Operator)**.** Define a linear differential operator as

$$L[y] = y'' + p(t)y' + q(t)y \tag{3.18}$$

where p and q are continuous on some open interval I where $t = \infty$ or both are included. The domain of this operator is the set of all twice differentiable functions. Its range is a set of unknown functions.

Thus a second order linear homogeneous differential equation is of the form $L[y] = 0$ and a second order linear non-homogeneous ODE is of the form $L[y] = g(t)$. Now consider an IVP

$$L[y] = 0, \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0' \tag{3.19}$$

There is an existence and uniqueness theorem for second order linear ODEs.

Theorem 3.1 (Existence and Uniqueness)**.** Consider the IVP $L[y] = g(t),\ y(t_0) = y_0$ and $y'(t_0) = y_0'$ where p, q and g are continuous on an open interval I that contains t_0. Then, there is exactly one solution $y(t) = \phi(t)$ and the solution exists throughout I.

Remark. The theorem tells us that the IVP has a solution, the solution is unique, and the solution is defined throughout the interval I where the coefficients are continuous, and the solution is at least twice differentiable.

Example 4

For the IVP $(t-1)y'' - 3ty + 4y = \sin t,\ y(-2) = 2$ and $y'(-2) = 1$, determine the largest interval where the IVP is certain to have a unique twice differentiable solution.

Solution

$$y'' - \frac{3t}{(t-1)}y + \frac{4}{(t-1)}y = \frac{\sin t}{t-1}$$

$\therefore t \neq 1 \implies (-\infty, 1)$ and $(1, \infty)$ are solution intervals. We choose $(-\infty, 1)$ as it includes the IC.

We know that the principle of superposition works for constant coefficients, but it turns out they work for any homogeneous linear second order ODE.

Theorem 3.2 (Principle of Superposition)**.** If y_1 and y_2 are solutions to $L[y] = y'' + p(t)y' + q(t)y = 0$, then any linear combination of y_1 and y_2 is also a solution, i.e.

$$y(t) = c_1 y_1 + c_2 y_2 \tag{3.20}$$

is a solution.

Proof.

$$
\begin{aligned}
L[c_1y_1 + c_2y_2] &= (c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2) && (3.21)\\
&= c_1y_1'' + c_2y_2'' + p(t)(c_1y_1' + c_2y_2') + q(t)(c_1y_1 + c_2y_2) && (3.22)\\
&= c_1[y_1'' + p(t)y_1' + q(t)y_1] + c_2[y_2'' + p(t)y_2' + q(t)y_2] && (3.23)\\
&= c_1L[y_1] + c_2L[y_2] = 0 && (3.24)
\end{aligned}
$$

■

However, we want to know if $y(t) = c_1y_1 + c_2y_2$ spans all possible solutions to our ODE. If we have the following initial conditions

$$
\begin{aligned}
y(t_0) &= y_0 && (3.25)\\
y'(t_0) &= y_0' && (3.26)
\end{aligned}
$$

Substitute these into our general solution to get

$$
\begin{aligned}
c_1y_1(t_0) + c_2y_2(t_0) &= y_0 && (3.27)\\
c_1y_1'(t_0) + c_2y_2'(t_0) &= y_0' && (3.28)
\end{aligned}
$$

By Cramer's rule (see MATH 416 notes), the solutions for c_1 and c_2 are

$$
c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \tag{3.29}
$$

For c_1 and c_2 to exist, the denominator, which is the same for both, must be non-zero. We define the denominator as the *Wronskian* evaluated at a point t_0.

Definition 3.2 (Wronskian)**.**

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \tag{3.30}$$

The Wronskian need not be evaluated at a point; it can be left in terms of y_1 and y_2 as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \tag{3.31}$$

From the Wronskian we get the important result.

Theorem 3.3. Suppose y_1 and y_2 are solutions of the second order linear ODE $L[y] = y'' + p(t)y' + q(t)y = 0$. Then, any linear combination

$$y = c_1y_1 + c_2y_2 \tag{3.32}$$

includes every solution of the ODE (ie. spans all possible solutions) if and only if there is a point t_0 where $W(y_1, y_2)(t_0) \neq 0$. In other words, if we change our initial conditions, we can always find a c_1 and c_2 that satisfies the equation.

Proof. Forwards. Assume our IC is at t_0 ie $y(t_0) = y_0$ and $y'(t_0) = y_0'$. Then substitute these into our general solution to get

$$c_1y_1(t_0) + c_2y_2(t_0) = y_0 \tag{3.33}$$

$$c_1y_1'(t_0) + c_2y_2'(t_0) = y_0' \tag{3.34}$$

By Cramer's rule, the solutions for c_1 and c_2 are the ratio of determinants shown above. Since $W(y_1, y_2)(t_0) \neq 0$, c_1 and c_2 exist. By the existence and uniqueness theorem, a solution of the ODE is uniquely determined by $y(t_0)$ and $y'(t_0)$. Thus, every solution arises this way.

Backwards. If $c_1y_1 + c_2y_2$ contains every solution, then there must also be c_1 and c_2

for the IVP $y(t_0) = 0$ and $y'(t_0) = 1$. This gives

$$c_1 = \frac{-y_2(t_0)}{W(y_1, y_2)(t_0)}, \qquad c_2 = \frac{y_1(t_0)}{W(y_1, y_2)(t_0)} \tag{3.35}$$

Thus, this exists only when $W(y_1, y_2)(t_0) \neq 0$. ■

This theorem says that the Wronskian of y_1 and y_2 is not zero everywhere if and only if the linear combination $c_1 y_1 + c_2 y_2$ contains all the solutions of the linear homogeneous ODE. Thus we call $y(t) = c_1 y_1(t) + c_2 y_2(t)$ the general solution of the linear homogeneous ODE, and we say y_1 and y_2 form a *fundamental set of solutions* to the ODE. This means that any solution to the ODE can be expressed as a linear combination of y_1 and y_2 (they act like basis vectors).

Thus to find the general solution of an ODE, we only need to find two solutions such that their Wronskian is non-zero. Then any linear combination of these solutions is a valid solution. We can now show that any differential equation of the form $L[y] = y'' + p(t)y' + q(t)y = 0$ always has a fundamental set of solutions.

Theorem 3.4. Consider the ODE $L[y] = y'' + p(t)y' + q(t)y = 0$ where p and q are continuous on some open interval I. Let y_1 be a solution that satisfies $y_1(t_0) = 1$ and $y_1'(t_0) = 0$ and y_2 be a solution that satisfies $y_2(t_0) = 0$ and $y_2'(t_0) = 1$. Then y_1 and y_2 form a fundamental set of solutions.

Proof. Since p and q are continuous, and y_1 and y_2 are at least twice differentiable, then y_1 and y_2 are guaranteed to exist by the existence and uniqueness theorem. To show y_1 and y_2 form a fundamental set of solutions we only need to compute $W(y_1, y_2)(t_0)$:

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \tag{3.36}$$

Since $W(y_1, y_2)(t_0) \neq 0$, y_1 and y_2 form a fundamental set of solutions. ■

We say that two functions f and g are *linearly independent* on an interval I if

$$k_1 f + k_2 g = 0 \implies k_1 = k_2 = 0 \tag{3.37}$$

In other words, neither is a scalar multiple of the other. Two functions are linearly dependent if they are not linearly independent. We can use the Wronskian to study the linear independence of functions.

Theorem 3.5. If f and g are differentiable functions on an open interval I and if $W(f,g) \neq 0$ for some $t_0 \in I$, then f and g are linearly independent on I. Moreover, if f and g are linearly dependent on I, then $W(f,g) = 0 \; \forall t \in I$.

Proof. For contradiction assume there exists constants c_1 and c_2 (not both zero) where

$$c_1 f(t) + c_2 g(t) = 0 \tag{3.38}$$

Evaluate at t_0, then differentiate and evaluate again,

$$c_1 f(t_0) + c_2 g(t_0) = 0 \tag{3.39}$$

$$c_1 f'(t_0) + c_2 g'(t_0) = 0 \tag{3.40}$$

This has coefficient matrix

$$\begin{bmatrix} f(t_0) & g(t_0) \\ f'(t_0) & g'(t_0) \end{bmatrix} \tag{3.41}$$

whose determinant is $W(f,g) \neq 0$. Thus the unique solution is $c_1 = c_2 = 0$ since this is the unique solution to a 2×2 system with non-zero determinant. Thus $W(f,g) \neq 0$ implies f and g are linearly independent.

Now we prove if f and g are linearly dependent then $W(f,g) = 0$. Assume f and g are linearly dependent on I. Then there exists constants c_1 and c_2 (not both 0) such that $c_1 f(t) + c_2 g(t) = 0$ on I. WLOG if $c_2 = 0$ then $c_1 \neq 0$ and $f = 0$. Thus $W = fg' - f'g = 0$.

If $c_2 \neq 0$ then $g = kf$ where $k = -c_1/c_2$. Thus,

$$W(f,g) = fg' - f'g = fkf' - f'kf = 0 \tag{3.42}$$

∎

One interesting thing about second order linear homogeneous ODEs is that even if we do not know the solutions of the ODE, we can still compute the Wronskian of any two solutions via Abel's theorem.

Theorem 3.6 (Abel's Theorem)**.** If y_1 and y_2 are solutions of the second order linear homogeneous ODE $L[y] = y'' + p(t)y' + q(t)y = 0$ where p and q are continuous on an open interval I, then the Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = ce^{-\int p(t)\,dt} \tag{3.43}$$

where c is a constant that depends on y_1 and y_2 but not on t. Further, $W(y_1, y_2)(t)$ is either 0 $\forall t \in I$ (if $c = 0$) or is never 0 in I (if $c \neq 0$).

Proof. Remember that y_1 and y_2 satisfy

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \tag{3.44}$$

$$y_2'' + p(t)y_2' + q(t)y_2 = 0 \tag{3.45}$$

Multiply the first equation by $-y_2$ and the second equation by y_1 and add them,

$$-y_2y_1'' - p(t)y_2y_1' - q(t)y_2y_1 + y_1y_2'' + p(t)y_1y_2' + q(t)y_1y_2 = 0 \tag{3.46}$$

$$\implies (y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0 \tag{3.47}$$

Let $W = W(y_1, y_2)(t)$ and observe $W' = y_1 y_2'' - y_1'' y_2$. Thus,

$$W' + p(t)W = 0 \tag{3.48}$$

$$\implies \frac{W'}{W} = -p(t) \tag{3.49}$$

Thus $[\ln |W|]' = -p(t)$,

$$\ln |W| = -\int p(t)\,dt + c_1 \implies W = ce^{-\int p(t)\,dt} \tag{3.50}$$

Since the exponential function is never 0, $W = 0$ iff $c = 0$ and if $c \neq 0$ then $W \neq 0$. ■

3.3 Complex Roots of the Characteristic Equation

For $ay'' + by' + cy = 0$ the characteristic polynomial is

$$ar^2 + br + c = 0 \implies r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.51}$$

If $b^2 - 4ac < 0$ then

$$r = -\frac{b}{2a} \pm \frac{i}{2a}\sqrt{4ac - b^2} \tag{3.52}$$

Define λ and μ by

$$\lambda = -\frac{b}{2a}, \qquad \mu = \frac{\sqrt{4ac - b^2}}{2a} \tag{3.53}$$

Then $r = \lambda \pm i\mu$. Our solutions are then

$$y_1(t) = e^{(\lambda + i\mu)t}, \qquad y_2(t) = e^{(\lambda - i\mu)t} \tag{3.54}$$

We can use Euler's formula to write this as a sum of sinusoids and work towards writing $y(t)$ as a real-valued function,

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{3.55}$$

3.3.0.1 Proof of Euler's formula using differential equations.

Consider the ODE $y''+y=0$. We make an ansatz (guess) that the solutions are $y_1 = \cos t$ and $y_2 = \sin t$,

$$y_1 = \cos t \implies y_1' = -\sin t \implies y_1'' = -\cos t \text{ (satisfies the ODE)} \tag{3.56}$$

$$y_2 = \sin t \implies y_2' = \cos t \implies y_2'' = -\sin t \text{ (satisfies the ODE)} \tag{3.57}$$

We now check if y_1 and y_2 are linearly independent,

$$W(y_1, y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \tag{3.58}$$

Since $W(y_1, y_2) \neq 0\ \forall t$, $\cos t$ and $\sin t$ are a fundamental set of solutions, ie. any solution to the ODE is a linear combination of $\cos t$ and $\sin t$. We also see $y = e^{it}$ is a solution to the ODE,

$$y = e^{it} \implies y' = ie^{it} \implies y'' = i^2 e^{it} = -e^{it} \text{ (so it satisfies)} \tag{3.59}$$

Thus e^{it} must be a linear combination of $\sin t$ and $\cos t$ ie

$$e^{it} = c_1 \cos t + c_2 \sin t \tag{3.60}$$

At $t = 0$, $e^0 = c_1 \cos(0) + c_2 \sin(0) = c_1 = 1$. Differentiate,

$$ie^{it} = -c_1 \sin t + c_2 \cos t \tag{3.61}$$

At $t = 0$, $i = c_2 \implies c_2 = i$. Thus, $e^{it} = \cos t + i \sin t$. ■

Since our solutions are $y_1 = e^{(\lambda+i\mu)t}$ and $y_2 = e^{(\lambda-i\mu)t}$, we can expand and use Euler's formula,

$$y_1 = e^{\lambda t + i\mu t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}[\cos(\mu t) + i \sin(\mu t)] \tag{3.62}$$

$$y_2 = e^{\lambda t - i\mu t} = e^{\lambda t} e^{-i\mu t} = e^{\lambda t}[\cos(\mu t) - i \sin(\mu t)] \tag{3.63}$$

Since y_1 and y_2 are solutions, any linear combination of y_1 and y_2 must also be a solution by the superposition of linear homogeneous ODEs,

$$u(t) = \frac{y_1 + y_2}{2} = e^{\lambda t}\cos(\mu t) \tag{3.64}$$

$$v(t) = \frac{y_1 - y_2}{2i} = e^{\lambda t}\sin(\mu t) \tag{3.65}$$

We can confirm if u and v form a fundamental set of solutions,

$$W(u,v) = \begin{vmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t) & \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t) \end{vmatrix} \tag{3.66}$$

$$= \left(\lambda e^{2\lambda t}\sin(\mu t)\cos(\mu t) + \mu e^{2\lambda t}\cos^2(\mu t)\right) - \left(\lambda e^{2\lambda t}\sin(\mu t)\cos(\mu t) - \mu e^{2\lambda t}\sin^2(\mu t)\right) \tag{3.67}$$

$$= \mu e^{2\lambda t}(\cos^2(\mu t) + \sin^2(\mu t)) \tag{3.68}$$

$$= \mu e^{2\lambda t} \tag{3.69}$$

Since $\mu = \sqrt{4ac - b^2}/(2a) \neq 0$, $\mu e^{2\lambda t} \neq 0 \ \forall t$. Thus $u(t)$ and $v(t)$ form a fundamental set of solutions. Thus the general solution to our ODE is

$$y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t) \tag{3.70}$$

In other words the solution to $ay'' + by' + cy = 0$ when the characteristic polynomial has complex roots is

$$y(t) = c_1 e^{-bt/(2a)}\cos\left(\frac{\sqrt{4ac - b^2}}{2a}t\right) + c_2 e^{-bt/(2a)}\sin\left(\frac{\sqrt{4ac - b^2}}{2a}t\right) \tag{3.71}$$

To plot a solution of the form $Ae^{\lambda t}\cos(\mu t) + Be^{\lambda t}\sin(\mu t)$, first factor to get

$$e^{\lambda t}(A\cos(\mu t) + B\sin(\mu t)) \tag{3.72}$$

Multiply by $\sqrt{A^2+B^2}/\sqrt{A^2+B^2}$ to get

$$\sqrt{A^2+B^2}e^{\lambda t}\left(\frac{A}{\sqrt{A^2+B^2}}\cos(\mu t)+\frac{B}{\sqrt{A^2+B^2}}\sin(\mu t)\right) \tag{3.73}$$

We know that $|A|<\sqrt{A^2+B^2}$ and $|B|<\sqrt{A^2+B^2}$ so

$$-1<\frac{A}{\sqrt{A^2+B^2}}<1, \qquad -1<\frac{B}{\sqrt{A^2+B^2}}<1 \tag{3.74}$$

Additionally,

$$\left(\frac{A}{\sqrt{A^2+B^2}}\right)^2+\left(\frac{B}{\sqrt{A^2+B^2}}\right)^2=1 \tag{3.75}$$

Thus we can define

$$\cos\alpha=\frac{A}{\sqrt{A^2+B^2}}, \qquad \sin\alpha=\frac{B}{\sqrt{A^2+B^2}} \tag{3.76}$$

Thus we now have

$$\sqrt{A^2+B^2}e^{\lambda t}\left(\cos(\mu t)\cos\alpha+\sin(\mu t)\sin\alpha\right) \tag{3.77}$$

which simplifies to

$$\sqrt{A^2+B^2}e^{\lambda t}\cos(\mu t-\alpha) \tag{3.78}$$

where $\tan\alpha=B/A$. Define $R=\sqrt{A^2+B^2}$. To plot $Re^{\lambda t}\cos(\mu t-\alpha)$, first plot $\pm Re^{\lambda t}$ as this envelopes the cosine. The plot of $Re^{\lambda t}\cos(\mu t-\alpha)$ lies within these two graphs.

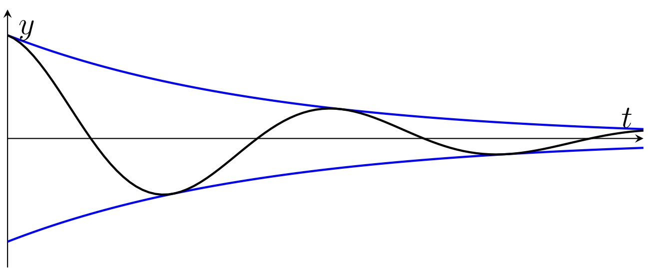


Figure 7: Envelope $\pm Re^{\lambda t}$ bounding $Re^{\lambda t}\cos(\mu t-\alpha)$.

The zeros of this curve occur at the zeros of $\cos(\mu t-\alpha)$ which is a periodic function and are hence equally spaced. The function touches the blue curves when $\cos(\mu t-\alpha)=\pm 1$, which are also equally spaced due to the periodicity of cosine.

3.4 Repeated Roots of the Characteristic Equation

Consider the ODE $ay'' + by' + cy = 0$ which has the characteristic polynomial

$$ar^2 + br + c = 0 \implies r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.79}$$

If $b^2 - 4ac = 0$, we have a case of repeated roots ie $r = -b/(2a)$. Both roots yield the same solution

$$y_1(t) = Ce^{-bt/(2a)} \tag{3.80}$$

We can't write this as $c_1 e^{-bt/(2a)} + c_2 e^{-bt/(2a)}$ since these are not linearly independent and won't form a fundamental set of solutions. We need a second solution so we have two linearly independent solutions so that we can span all possible solutions of the ODE.

d'Alembert proposed there is a second solution of the form

$$y(t) = v(t)y_1(t) \tag{3.81}$$

Thus,

$$y' = v'(t)y_1(t) + v(t)y_1'(t) \tag{3.82}$$

$$y'' = v''(t)y_1(t) + v'(t)y_1'(t) + v'(t)y_1'(t) + v(t)y_1''(t) \tag{3.83}$$

$$= v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t) \tag{3.84}$$

$$= v''y_1 + 2v'y_1' + vy_1'' \tag{3.85}$$

Substituting into $ay'' + by' + cy = 0$,

$$a(v''y_1 + 2v'y_1' + vy_1'') + b(v'y_1 + vy_1') + cvy = 0 \tag{3.86}$$

Since $y_1 = e^{-bt/(2a)}$, $y_1' = -\frac{b}{2a}e^{-bt/(2a)} = -\frac{b}{2a}y_1$. Expanding the ODE,

$$2av'y_1' + bv'y_1 = 2av'\left(-\frac{b}{2a}y_1\right) + bv'y_1 \tag{3.87}$$

$$= -bv'y_1 + bv'y_1 \tag{3.88}$$

$$= 0 \tag{3.89}$$

Thus,

$$av''y_1 + avy_1'' + bvy_1' + cvy = 0 \tag{3.90}$$

$$\implies av''y_1 + v(ay_1'' + by_1' + cy_1) = 0 \tag{3.91}$$

Since y_1 is a solution we have

$$av''y_1 = 0 \implies v'' = 0 \text{ (since } a \neq 0,\ y_1 > 0) \tag{3.92}$$

Thus $v = b_1 t + b_2$. Thus the second solution $y_2(t)$ is of the form

$$y_2(t) = te^{-bt/(2a)} \tag{3.93}$$

We can confirm linear independence via the Wronskian,

$$W(y_1, y_2) = \begin{vmatrix} e^{-bt/(2a)} & te^{-bt/(2a)} \\ -\frac{b}{2a}e^{-bt/(2a)} & e^{-bt/(2a)} - \frac{b}{2a}te^{-bt/(2a)} \end{vmatrix} \tag{3.94}$$

$$= e^{-bt/a} - \frac{b}{2a}te^{-bt/a} + \frac{b}{2a}te^{-bt/a} \tag{3.95}$$

$$= e^{-bt/a} \tag{3.96}$$

Thus $W(y_1, y_2) \neq 0\ \forall t$ so they are linearly independent and form a fundamental set of solutions. Thus for the ODE $ay''+by'+cy = 0$ with characteristic polynomial $ar^2+br+c =$

0 that has repeated roots (ie $b^2 - 4ac = 0$), the general solution is

$$y(t) = c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)} \tag{3.97}$$

Example 5

$y'' - \frac{2}{3}y' + \frac{1}{9}y = 0$, $y(0) = 3$ and $y'(0) = b$. Find the value of b that separates solutions that grow positively vs those that initially grow negatively.

Solution

$$r^2 - \frac{2}{3}r + \frac{1}{9} = 0 \implies \left(r - \frac{1}{3}\right)^2 = 0 \implies r = \frac{1}{3}$$

$$y(t) = c_1 e^{t/3} + c_2 t e^{t/3}$$

Using $y(0) = 3$,

$$y(0) = c_1 = 3$$

Using $y'(0) = b$, $y'(t) = \frac{c_1}{3}e^{t/3} + c_2 e^{t/3} + \frac{c_2}{3}te^{t/3}$,

$$y'(0) = \frac{c_1}{3} + c_2 = 1 + c_2 = b \implies c_2 = b - 1$$

Thus

$$y(t) = 3e^{t/3} + (b-1)te^{t/3} = e^{t/3}(3 + (b-1)t)$$

If $b - 1 > 0$ then $b > 1$ so solution grows positively. If $b - 1 < 0$ then $3 - (1-b)t < 0 \iff t > 3/(1-b)$. Value of b that separates is $b = 1$.

3.4.0.1 Summary for homogeneous second-order homogeneous differential equations.

For the second-order ODE $ay'' + by' + cy = 0$ with characteristic polynomial $ar^2 + br + c = 0 \implies r = (-b \pm \sqrt{b^2 - 4ac})/(2a)$:

- If $b^2 - 4ac > 0$ (two real roots) then

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac = 0$ (repeated roots) then

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}, \quad r = -\frac{b}{2a}$$

- If $b^2 - 4ac < 0$ (complex roots) then

$$y(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t), \quad \lambda = -\frac{b}{2a}, \ \mu = \frac{\sqrt{4ac - b^2}}{2a}$$

3.5 Reduction of Order

Suppose we have a linear homogeneous second order ODE given by

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{3.98}$$

If we know one solution $y_1(t)$ which is not zero everywhere, then by d'Alembert's method for repeated roots, the second solution is

$$y_2(t) = v(t)y_1(t) \tag{3.99}$$

Substitute y_2 into the ODE,

$$y_2' = v'y_1 + vy_1' \tag{3.100}$$

$$y_2'' = v''y_1 + v'y_1' + v'y_1' + vy_1'' = v''y_1 + 2v'y_1' + vy_1'' \tag{3.101}$$

$$L[y_2] = v''y_1 + 2v'y_1' + vy_1'' + p(t)[v'y_1 + vy_1'] + q(t)[vy_1] \tag{3.102}$$

$$= y_1 v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0 \tag{3.103}$$

Since $L[y_1] = 0$ we have

$$y_1 v'' + (2y_1' + py_1)v' = 0 \tag{3.104}$$

Define $u = v'$. Then we have

$$y_1 u' + (2y_1' + py_1)u = 0 \tag{3.105}$$

This is a first order ODE which we can solve to get $u = v'$, and then integrate to get v. Our second solution is then $y_2 = vy_1$.

3.6 Solving Non-Homogeneous Linear Second Order ODEs

Consider a linear non-homogeneous ODE of the form

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{3.106}$$

where p, q and g are continuous functions on an open interval I. We define the corresponding homogeneous equation of the ODE above as the following ODE

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{3.107}$$

Theorem 3.7. If Y_1 and Y_2 are solutions of the non-homogeneous ODE then $Y_1 - Y_2$ is a solution to the corresponding homogeneous ODE. Also if y_1 and y_2 form a fundamental set of solutions to the homogeneous ODE then

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2 \tag{3.108}$$

Proof.

$$
\begin{aligned}
L[Y_1 - Y_2] &= (Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(t)(Y_1 - Y_2) && (3.109)\\
&= [Y_1'' + p(t)Y_1' + q(t)Y_1] - [Y_2'' + p(t)Y_2' + q(t)Y_2] && (3.110)\\
&= L[Y_1] - L[Y_2] && (3.111)\\
&= g(t) - g(t) = 0 && (3.112)
\end{aligned}
$$

Thus $Y_1 - Y_2$ satisfies the corresponding homogeneous ODE. Also, since y_1 and y_2 form a fundamental set of solutions to the ODE, any solution can be expressed as a linear combination of y_1 and y_2. ■

Thus, the complementary solution is needed to span all possible solutions of our ODE. There are two ways to solve for the particular solution:

1. *Undetermined coefficients:* You try and guess what the general solution looks like and substitute it into the ODE to determine the coefficients that satisfies the equation. It is extremely easy but only works with constant coefficients and when $g(t)$ is an exponential function, polynomial or sinusoidal.

2. *Variation of parameters:* Replace the constants in the corresponding homogeneous equation with functions $u(t)$ and $v(t)$, substitute into the ODE and solve for u and v. Can be used whenever but requires integration to get u and v which does not always have an analytic solution.

Theorem 3.8. The general solution of the non-homogeneous ODE can be expressed as

$$y = \phi(t) = c_1y_1 + c_2y_2 + Y \tag{3.113}$$

where y_1 and y_2 are solutions of the corresponding homogeneous ODE and Y is a solution to the non-homogeneous ODE.

Proof. Define $Y_1 = \phi$ and $Y_2 = Y$. Then

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2 \tag{3.114}$$

$$\implies \phi - Y = c_1 y_1 + c_2 y_2 \tag{3.115}$$

$$\implies \phi(t) = c_1 y_1 + c_2 y_2 + Y \tag{3.116}$$

■

The $c_1 y_1 + c_2 y_2$ term is known as the *complementary solution* y_c and Y is known as the *particular solution* y_p. Thus the solution to the linear non-homogeneous second order ODE is

$$y = \phi(t) = y_c + y_p = c_1 y_1 + c_2 y_2 + Y \tag{3.117}$$

The reason this works is because

$$L[\phi] = L[y_c + y_p] = L[y_c] + L[y_p] \tag{3.118}$$

But $L[y_c] = 0$ so $L[\phi] = L[y_p] = g(t)$.

3.7 Method of Undetermined Coefficients

The method of undetermined coefficients requires us to guess what the solution to the non-homogeneous ODE looks like after we solve for the complementary solution. Depending on what $g(t)$ is, we make a guess for $Y(t)$ and determine its coefficients. If we do not get valid coefficients then we may not have a valid solution to the ODE. Here are valid guesses.

$g(t)$	$Y(t)$
$P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_0)$
$P_n(t) e^{\alpha t}$	$t^s Q_n(t) e^{\alpha t}$
$R \sin(\beta t)$ or $R \cos(\beta t)$	$A \sin(\beta t) + B \cos(\beta t)$
$P_n(t) e^{\alpha t} \sin(\beta t)$ or $P_n(t) e^{\alpha t} \cos(\beta t)$	$t^s e^{\alpha t}(Q_n(t) \cos \beta t + M_n(t) \sin \beta t)$

$s \geq 0$ is the lowest degree that ensures no terms in $Y(t)$ are solutions of the corresponding homogeneous ODE, as then we would not solve the non-homogeneous ODE.

Example 6

Find the general solution to $y'' + 4y = t^2 + 3e^t$.

Solution

The corresponding homogeneous ODE is $y'' + 4y = 0$ which has characteristic polynomial $r^2+4 = 0 \implies r = \pm 2i$ so $\lambda = 0$ and $\mu = 2$. Thus the complementary solution is

$$c_1 e^{0t}\cos(2t) + c_2 e^{0t}\sin(2t) = c_1\cos(2t) + c_2\sin(2t)$$

Now guess that the solution to the non-homogeneous ODE is

$$Y = at^2 + bt + c + de^t$$

$$Y' = 2at + b + de^t, \qquad Y'' = 2a + de^t$$

$$Y'' + 4Y = 2a + de^t + 4at^2 + 4bt + 4c + 4de^t = t^2 + 3e^t$$

$$\implies 4at^2 + 4bt + (2a + 4c) + 5de^t = t^2 + 3e^t$$

Thus $4a = 1, 5d = 3, 4b = 0, 2a + 4c = 0$, so

$$a = \frac{1}{4},\ d = \frac{3}{5},\ b = 0,\ 4c = -2a \implies c = -\frac{a}{2} = -\frac{1}{8}$$

Thus,

$$Y = \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$$

Thus the general solution is

$$y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$$

Example 7

Find the general solution to $y'' + 9y = \sin(3t)$.

Solution

$r^2 + 9 = 0 \implies r = \pm 3i$ so $y_c = c_1 \cos(3t) + c_2 \sin(3t)$. Since $g(t)$ is the same as y_2, for our guess we need to multiply by t ($s = 1$) to avoid duplicating terms with the complementary solution,

$$Y = t[A\cos(3t) + B\sin(3t)]$$

$$Y' = A\cos(3t) + B\sin(3t) + t[3B\cos(3t) - 3A\sin(3t)]$$

$$Y'' = 3B\cos(3t) - 3A\sin(3t) + 3B\cos(3t) - 3A\sin(3t) + t[-9B\sin(3t) - 9A\cos(3t)]$$

$$= 6B\cos(3t) - 6A\sin(3t) - 9t[B\sin 3t + A\cos 3t]$$

$$Y'' + 9Y = 6B\cos(3t) - 6A\sin(3t) = \sin(3t)$$

$\therefore B = 0$ and $A = -\frac{1}{6}$. Thus,

$$y(t) = c_1\cos(3t) + c_2\sin(3t) - \frac{1}{6}t\sin(3t)$$

If we have something like $L[y] = g_1(t) + g_2(t)$ we can solve $L[y] = g_1(t)$ and $L[y] = g_2(t)$ separately and simply add their particular solutions (they have the same complementary solution).

Example 8

$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t\cos(2t)$.

Solution

$r^2 - 3r - 4 = 0 \implies (r-4)(r+1) = 0 \implies r = 4$ or $r = -1$. $\therefore y_c = c_1 e^{4t} + c_2 e^{-t}$.

For particular solutions solve separately, $y'' - 3y' - 4y = 3e^{2t}$, $y'' - 3y' - 4y = 2\sin t$, $y'' - 3y' - 4y = -8e^t \cos(2t)$. Then the general solution is

$$y(t) = c_1 e^{4t} + c_2 e^{-t} + y_{p_1} + y_{p_2} + y_{p_3}$$

3.8 Method of Variation of Parameters

Unlike undetermined coefficients, variation of parameters can be used to solve any equation regardless of the form of $g(t)$. However, this may involve integration which is not always analytic. For $L[y] = y'' + p(t)y' + q(t)y = g(t)$, the complementary solution to the corresponding homogeneous equation is

$$y_c = c_1 y_1 + c_2 y_2 \tag{3.119}$$

Similar to reduction of order, for the particular solution we replace c_1 and c_2 with $u_1(t)$ and $u_2(t)$,

$$y_p = u_1 y_1 + u_2 y_2 \tag{3.120}$$

Now

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2' \tag{3.121}$$

We already see if we substitute into our ODE we have one equation and two unknowns (u_1 and u_2). Since we artificially introduced u_1 and u_2, we can artificially set constraints on it. The easiest constraint is to set

$$u_1' y_1 + u_2' y_2 = 0 \tag{3.122}$$

Thus,

$$y_p' = u_1 y_1' + u_2 y_2' \tag{3.123}$$

$$\implies y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' \tag{3.124}$$

Substitute into our ODE,

$$L[y_p] = y_p'' + p(t) y_p' + q(t) y_p \tag{3.125}$$

$$= u_1(y_1'' + p(t) y_1' + q(t) y_1) + u_2(y_2'' + p(t) y_2' + q(t) y_2) + u_1' y_1' + u_2' y_2' \tag{3.126}$$

$$= g(t) \tag{3.127}$$

Since $L[y_1] = L[y_2] = 0$ (they solve the corresponding homogeneous equation) we have

$$u_1' y_1' + u_2' y_2' = g(t) \tag{3.128}$$

$$u_1' y_1 + u_2' y_2 = 0 \tag{3.129}$$

Multiply the first by y_2 and second by y_2' to get

$$u_1' y_1' y_2 + u_2' y_2' y_2 = y_2 g \tag{3.130}$$

$$u_1' y_1 y_2' + u_2' y_2 y_2' = 0 \tag{3.131}$$

Thus,

$$u_1'(y_1' y_2 - y_1 y_2') = y_2 g \tag{3.132}$$

$$\implies u_1'(-W(y_1, y_2)) = y_2 g \tag{3.133}$$

Recall $W(y_1, y_2) = y_1 y_2' - y_1' y_2$, so

$$u_1' = -\frac{y_2 g}{W(y_1, y_2)} \implies u_1(t) = \int -\frac{y_2(t) g(t)}{W(y_1, y_2)}\, dt + c_1 \tag{3.134}$$

Now $u_1' y_1 + u_2' y_2 = 0 \implies u_2' = -u_1' y_1/y_2$. Thus,

$$u_2' = \frac{y_1 g}{W(y_1, y_2)} \implies u_2(t) = \int \frac{y_1(t) g(t)}{W(y_1, y_2)}\, dt + c_2 \tag{3.135}$$

Thus, the general solution to the inhomogeneous ODE is

$$y(t) = y_c + y_p \tag{3.136}$$

which gives

$$y(t) = c_1 y_1 + c_2 y_2 + y_1 \left(\int -\frac{y_2 g}{W(y_1, y_2)}\, dt + b_1 \right) + y_2 \left(\int \frac{y_1 g}{W(y_1, y_2)}\, dt + b_2 \right) \tag{3.137}$$

It is always better to try and derive than memorise.

Example 9

Solve $y'' + y' - 6y = e^t$ via variation of parameters.

Solution

$r^2+r-6=0 \implies (r+3)(r-2)=0 \implies r=-3$ and $r=2$. $\therefore y_c = c_1e^{-3t}+c_2e^{2t}$.

$$y_p = ue^{-3t} + ve^{2t}$$

$$\implies y_p' = -3ue^{-3t} + u'e^{-3t} + 2ve^{2t} + v'e^{2t}$$

Set $u'e^{-3t} + v'e^{2t} = 0 \implies y_p' = -3ue^{-3t} + 2ve^{2t}$.

$$y_p'' = 9ue^{-3t} - 3u'e^{-3t} + 4ve^{2t} + 2v'e^{2t}$$

$$y_p'' + y_p' - 6y_p = -3u'e^{-3t} + 2v'e^{2t} = e^t$$

Combined with $u'e^{-3t} + v'e^{2t} = 0$:

$$-3u'e^{-3t} + 2v'e^{2t} = e^t, \qquad 3u'e^{-3t} + 3v'e^{2t} = 0$$

Adding, $5v'e^{2t} = e^t \implies v' = \frac{1}{5}e^{-t}$. Thus

$$v = \int \frac{1}{5}e^{-t}\,dt = -\frac{1}{5}e^{-t} + b_1$$

Then $u'e^{-3t} = -v'e^{2t} = -\frac{1}{5}e^{-t}e^{2t} = -\frac{1}{5}e^t$, so $u' = -\frac{1}{5}e^{4t}$,

$$u = \int -\frac{1}{5}e^{4t}\,dt = -\frac{1}{20}e^{4t} + b_2$$

$\therefore y(t) = c_1y_1 + c_2y_2 + uy_1 + vy_2$

$$\begin{aligned}
&= c_1e^{-3t} + c_2e^{2t} + \left(-\frac{1}{20}e^{4t} + b_2\right)e^{-3t} + \left(-\frac{1}{5}e^{-t} + b_1\right)e^{2t} \\
&= c_1e^{-3t} + c_2e^{2t} - \frac{1}{20}e^t + b_2e^{-3t} - \frac{1}{5}e^t + b_1e^{2t} \\
&= a_1e^{-3t} + a_2e^{2t} - \frac{1}{4}e^t
\end{aligned}$$

3.9 Mechanical Vibrations

Second-order ODEs can be used to model the phenomena of mechanical vibrations — particularly the motion of a mass on a spring. Consider a spring of original length ℓ which extends by a length L when a mass m is attached to it. When the system reaches

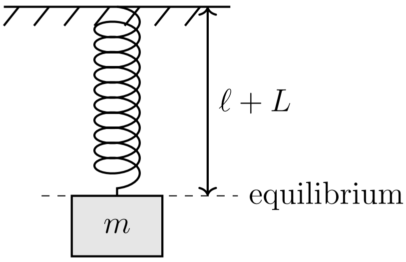


Figure 8: Mass on a spring at equilibrium.

equilibrium, Newton's second law gives us

$$F_w - F_s = 0 \tag{3.138}$$

$$\implies mg - kL = 0 \tag{3.139}$$

k is the spring constant of the spring and g is the acceleration due to gravity. We now displace the mass such that its position from its new equilibrium position (where downwards is positive) is given by $x(t)$. Then by Newton's second law again,

$$m\frac{d^2x}{dt^2} = f(t) \tag{3.140}$$

where $f(t)$ is the net force on the mass. The four forces that compose $f(t)$ are as follows:

1. Weight of the mass: $F_w = mg$ acting downward (+).
2. The spring force $F_s = -k(L + x(t))$ exerted on the spring. If $L + x(t) > 0$, the spring is extended from equilibrium and so the spring force is directed upwards ($-$). If $L+x(t) < 0$ the spring is compressed from equilibrium and the spring force is directed downwards (+).
3. Damping forces like air resistance F_d which always acts opposite the direction of motion

of the mass. This force is proportional to the velocity of the mass, giving way to a phenomena known as viscous damping. The force is given by $F_d(t) = -\gamma \frac{dx}{dt}$ where γ is a positive constant known as the damping constant.

4. Any applied external force $F(t)$ directed upwards or downwards either due to the motion of the mount of the spring or a force applied to the mass.

Substituting these forces into Newton's second law gives

$$m\frac{d^2x}{dt^2} = mg - k(L + x) - \gamma\frac{dx}{dt} + F(t) \tag{3.141}$$

Expanding,

$$m\frac{d^2x}{dt^2} = mg - kL - kx - \gamma\frac{dx}{dt} + F(t) \tag{3.142}$$

We know $mg - kL = 0$ by the original equilibrium position. Thus,

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = F(t) \tag{3.143}$$

We now specify two initial conditions: the original position and velocity of the mass,

$$x(0) = x_0, \qquad x'(0) = v_0 \tag{3.144}$$

Solving this IVP will give us the position of the mass as a function of time which we can then use to get the velocity and acceleration which allows us to fully understand the motion of the mass.

This equation is an approximation as the damping and spring force equations only hold true for small velocities/displacements. We have also neglected the mass of the spring/mount, etc. By the existence and uniqueness theorem, the IVP has a unique solution which is consistent with our physical intuition that if the mass is set in motion with an initial displacement and velocity, its position will be determined uniquely at all future times.

3.9.1 Undamped Free Vibrations

We often study undamped free vibrations, which occur when there is no damping force ($\gamma = 0$) and applied external force ($F(t) = 0$). The ODE then becomes

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{3.145}$$

This has characteristic polynomial

$$mr^2 + k = 0 \implies r = \pm\frac{\sqrt{-4km}}{2m} = \pm\frac{\sqrt{km}}{m}i = \pm\sqrt{\frac{k}{m}}i \tag{3.146}$$

We define the *natural frequency* of the system ω_0 as

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{3.147}$$

The natural frequency is the frequency at which the system oscillates when not subjected to a continuous external force. The solution of the undamped free ODE is then

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \tag{3.148}$$

As done in 3.3, multiply by $\sqrt{A^2+B^2}/\sqrt{A^2+B^2}$ to get

$$x(t) = R\cos(\omega_0 t - \phi) \tag{3.149}$$

so the mass oscillates sinusoidally. R is known as the amplitude of oscillation and ϕ is known as the phase, both given by

$$R = \sqrt{A^2+B^2}, \qquad \arctan\phi = \frac{B}{A} \tag{3.150}$$

To get the correct phase, we need to look at the sign of $\cos\phi = A/R$ and $\sin\phi = B/R$ and select the correct quadrant. The amplitude $R = \sqrt{A^2+B^2}$ is determined by the initial conditions. We call the motion of this system simple harmonic as it is periodic.

The period of one oscillation is given by

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}} \tag{3.151}$$

Since there is no damping involved, the mass will oscillate forever since energy from the system is not dissipated. The frequency of the oscillation is determined by the mass and spring constant (stiffer springs, i.e. higher k, cause more rapid oscillations).

3.9.2 Damped Free Vibrations

Damped free oscillations occur when there is no external applied force ($F(t) = 0$) but there is damping ($\gamma > 0$). The ODE then becomes

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0 \tag{3.152}$$

The characteristic polynomial is

$$mr^2 + \gamma r + k = 0 \tag{3.153}$$

This has roots

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} \tag{3.154}$$

Depending on the sign of the discriminant, the form changes.

1. $\gamma^2 - 4km > 0$ (overdamped motion): two distinct real roots. Since $\gamma, m, k > 0$, $\gamma^2 - 4km$ is always less than γ^2 so the roots are real negative. The solution is of the form

$$x(t) = c_1e^{r_1t} + c_2e^{r_2t} \tag{3.155}$$

2. $\gamma^2 - 4km = 0$ (critically damped motion): one real negative root with multiplicity 2. The solution is of the form

$$x(t) = c_1e^{-\gamma t/(2m)} + c_2te^{-\gamma t/(2m)} \tag{3.156}$$

3. $\gamma^2 - 4km < 0$ (underdamped motion): two complex conjugate roots with negative real part. The solution is of the form

$$x(t) = c_1 e^{-\gamma t/(2m)} \cos(\mu t) + c_2 e^{-\gamma t/(2m)} \sin(\mu t) \tag{3.157}$$

where

$$\mu = \frac{\sqrt{4km - \gamma^2}}{2m} \tag{3.158}$$

μ is called the *quasi frequency* and determines the frequency at which the mass oscillates. We define the *quasi period* T_d as

$$T_d = \frac{2\pi}{\mu} \tag{3.159}$$

This determines the time between each oscillation.

Regardless of the type of damping, $x(t) \to 0$ as $t \to \infty$ which is because damping dissipates energy in the system. However, the type of damping does determine the way the system oscillates.

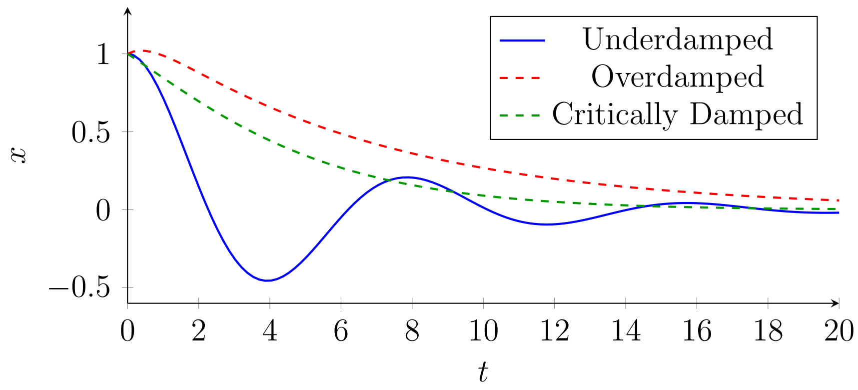


Figure 9: Comparison of underdamped, overdamped, and critically damped motion.

3.9.3 Undamped Forced Vibrations

If we have the same external applied force but no damping we get

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos(\omega t) \tag{3.160}$$

If $\omega \neq \omega_0$, the general solution is

$$x = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t) \tag{3.161}$$

This shows the motion is the sum of two periodic motions of different frequencies and different amplitudes.

3.9.4 Damped Forced Vibrations

In damped forced vibrations we assume there is both damping ($\gamma > 0$) and an external applied force ($F(t) \neq 0$). We assume the external force is of the form

$$F(t) = F_0 \cos(\omega t) \tag{3.162}$$

The overall ODE is then

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = F_0 \cos(\omega t) \tag{3.163}$$

The corresponding homogeneous equation is just the case of damped free vibrations; the solution to the homogeneous equation is known as the *transient solution.* We assume the particular solution is of the form

$$X = A\cos(\omega t) + B\sin(\omega t) \tag{3.164}$$

$$X' = -\omega A \sin(\omega t) + \omega B \cos(\omega t) \tag{3.165}$$

$$X'' = -\omega^2 A \cos(\omega t) - \omega^2 B \sin(\omega t) = -\omega^2(A\cos(\omega t) + B\sin(\omega t)) \tag{3.166}$$

Thus,

$$-m\omega^2(A\cos(\omega t) + B\sin(\omega t)) + \gamma(-A\omega\sin(\omega t) + B\omega\cos(\omega t)) \tag{3.167}$$

$$+ k(A\cos(\omega t) + B\sin(\omega t)) = F_0\cos(\omega t) \tag{3.168}$$

Thus, by comparing coefficients,

$$-m\omega^2 A + \gamma\omega B + kA = F_0 \tag{3.169}$$

$$-m\omega^2 B - \gamma\omega A + kB = 0 \tag{3.170}$$

This gives

$$A = \frac{-m\omega^2 B + kB}{\gamma\omega} \tag{3.171}$$

Substituting,

$$-m\omega^2 \left(\frac{-m\omega^2 B + kB}{\gamma\omega}\right) + \gamma\omega B + k\left(\frac{-m\omega^2 B + kB}{\gamma\omega}\right) = F_0 \tag{3.172}$$

Thus,

$$B\left(\frac{m^2\omega^3 - mk\omega + \gamma^2\omega^2 + k^2 - mk\omega^2}{\gamma\omega}\right) \cdot \frac{1}{\omega} = F_0 \tag{3.173}$$

$$\implies B\left(\frac{m^2\omega^4 - 2m\omega^2 k + k^2 + \gamma^2\omega^2}{\gamma\omega}\right) = F_0 \tag{3.174}$$

Define α as

$$\alpha = \sqrt{m^2\omega^4 - 2m\omega^2 k + k^2 + \gamma^2\omega^2} = \sqrt{(k - m\omega^2)^2 + (\gamma\omega)^2} \tag{3.175}$$

Thus,

$$B = \frac{F_0\gamma\omega}{\alpha^2} \tag{3.176}$$

which gives

$$A = \frac{F_0(k - m\omega^2)}{\alpha^2} \tag{3.177}$$

The particular solution is written as

$$U(t) = R\cos(\omega t - \phi) \tag{3.178}$$

where

$$R = \sqrt{A^2 + B^2} = \frac{F_0}{\alpha} \tag{3.179}$$

$$\cos\phi = \frac{k - m\omega^2}{\alpha} = \frac{m(\omega_0^2 - \omega^2)}{\alpha} \tag{3.180}$$

$$\sin\phi = \frac{\gamma\omega}{\alpha} \tag{3.181}$$

where $\omega_0^2 = k/m$. $U(t)$ does not die out as $t \to \infty$, thus it represents steady oscillations with the same frequency as the external force. Thus the particular solution $U(t)$ is called the *steady-state solution* or *forced response*.

4 Higher-Order Linear Differential Equations

4.1 General Theory of nth Order Linear ODEs

An nth order linear differential equation is of the form

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_n(t)y = G(t) \tag{4.1}$$

If we divide by $P_0(t)$ we can express our ODE as the linear differential operator of order n,

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_n(t)y = g(t) \tag{4.2}$$

For an nth order ODE, we need n initial conditions to create an IVP,

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)} \tag{4.3}$$

where $t_0 \in I = (\alpha, \beta)$ where the functions $p_i(t)$ for $i \in \{1, \ldots, n\}$ are continuous. The theory of nth order ODEs is analogous to that of second order ODEs.

Theorem 4.1 (Existence and Uniqueness)**.** If we have

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_n(t)y = g(t), \qquad y(t_0) = y_0,\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)} \tag{4.4}$$

and $p_1, \ldots, p_n$ are continuous on some interval $I = (\alpha, \beta)$ where $t_0 \in I$, then there exists a unique solution $y = \phi(t)$ that satisfies the IVP.

The next result relates the Wronskian and linearly independent solutions for nth order ODEs.

Theorem 4.2. For $\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_n(t)y = 0$ where $p_1, \ldots, p_n$ are continuous on an open interval $I = (\alpha, \beta)$. If $y_1, \ldots, y_n$ solve the homogeneous equation

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_n(t)y = 0 \tag{4.5}$$

and if $W(y_1, \ldots, y_n)(t) \neq 0$ for at least one point in I, then every solution of the homogeneous ODE can be expressed as a linear combination of $y_1, \ldots, y_n$,

$$y(t) = c_1 y_1 + \cdots + c_n y_n \tag{4.6}$$

A set of solutions $(y_1, \ldots, y_n)$ of the homogeneous ODE whose Wronskian is non-zero at a point is known as a *fundamental set of solutions.*

4.2 nth Order Homogeneous ODEs with Constant Coefficients

Consider the homogeneous nth order linear ODE with constant coefficients

$$L[y] = a_0 y^{(n)} + \cdots + a_n y = 0 \tag{4.7}$$

If we take $y = e^{rt}$ as our ansatz we get

$$e^{rt}(a_0 r^n + \cdots + a_n) = e^{rt}Z(r) = 0 \implies Z(r) = 0 \tag{4.8}$$

This is our *characteristic polynomial* whose roots will give our solutions. If we have real and unequal roots $r_1, \ldots, r_n$ our solution is of the form

$$y = c_1 e^{r_1 t} + \cdots + c_n e^{r_n t} \tag{4.9}$$

If we have complex roots, they must occur in conjugate pairs

$$r_1 = \lambda_1 \pm i\mu_1, \ \ldots, \ r_k = \lambda_k \pm i\mu_k \tag{4.10}$$

Our solution is of the form

$$y = e^{\lambda_1 t}(c_1 \cos(\mu_1 t) + c_2 \sin(\mu_1 t)) + \cdots + e^{\lambda_k t}(c_{2k-1} \cos(\mu_k t) + c_{2k} \sin(\mu_k t)) \tag{4.11}$$

If we have repeated roots with multiplicity $s \leq n$, the solutions are

$$y_1 = e^{r_1 t},\ y_2 = te^{r_1 t},\ y_3 = t^2 e^{r_1 t},\ \ldots,\ y_n = t^{s-1} e^{r_1 t} \tag{4.12}$$

If we have a complex root $\lambda \pm i\mu$ repeated s times, the complex conjugate must also be repeated s times. Thus,

$$y_1 = e^{\lambda t}(\cos(\mu t) + \sin(\mu t)) \tag{4.13}$$

$$y_2 = te^{\lambda t}(\cos(\mu t) + \sin(\mu t)) \tag{4.14}$$

$$\vdots \tag{4.15}$$

$$y_n = t^{s-1} e^{\lambda t}(\cos(\mu t) + \sin(\mu t)) \tag{4.16}$$

Example 10

Find the general solution to $y^{(4)} - 4y'' + 3y = 0$.

Solution

Characteristic polynomial: $r^4 - 4r^2 + 3 = 0$. Let $y = r^2$, then $y^2 - 4y + 3 = (y - 3)(y - 1) = (r^2 - 1)(r^2 - 3) = 0$. Thus $r^2 = 1$ or $r^2 = 3$, giving $r = \pm 1,\ r = \pm\sqrt{3}$. The general solution is

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{\sqrt{3}t} + c_4 e^{-\sqrt{3}t}$$

Example 11

Find the general solution to $y^{(4)} + 2y'' + y = 0$.

Solution

$r^4 + 2r^2 + 1 = 0 \implies (r^2+1)(r^2+1) = 0$. The roots are $\pm i$ with multiplicity 2, so we get

$$y(t) = c_1 \cos t + c_2 \sin t + t[c_3 \cos t + c_4 \sin t]$$

4.3 Undetermined Coefficients for nth Order ODEs

Undetermined coefficients for nth order ODEs is the exact same as it was for second order ODEs. You find the complementary solution y_c to the corresponding homogeneous ODE and make an ansatz for the particular solution y_p and solve for its coefficients. The solution is then

$$y = y_c + y_p \tag{4.17}$$

Example 12

Find the general solution to

$$y^{(4)} + 2y'' + y = 5 + \sin(2t)$$

Solution

Characteristic polynomial of the homogeneous ODE:

$$r^4 + 2r^2 + 1 = 0 \implies (r^2+1)(r^2+1) = 0 \implies r = \pm i \text{ (multiplicity 2)}$$

$\therefore y_c = c_1 \cos t + c_2 \sin t + t[c_3 \cos t + c_4 \sin t]$. Ansatz: $y = A\cos(2t) + B\sin(2t) + C$.

$$\begin{aligned}
y' &= -2A\sin(2t) + 2B\cos(2t) \\
y'' &= -4A\cos(2t) - 4B\sin(2t) \\
y''' &= 8A\sin(2t) - 8B\cos(2t) \\
y^{(4)} &= 16A\cos(2t) - 16B\sin(2t)
\end{aligned}$$

$$\begin{aligned}
\therefore y^{(4)} + 2y'' + y &= 16A\cos(2t) - 16B\sin(2t) - 8A\cos(2t) - 8B\sin(2t) + A\cos(2t) \\
&= 5 + \sin(2t)
\end{aligned}$$

$$\implies 9A\cos(2t) - 9B\sin(2t) + C = 5 + \sin(2t)$$

$$\implies A = 0,\ B = -\frac{1}{9},\ C = 5$$

$\therefore y_p = -\frac{1}{9}\sin(2t) + 5$,

$$y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t - \frac{1}{9}\sin(2t) + 5$$

Example 13

Find a form for the particular solution of

$$y^{(4)} - y''' + 2y'' + 4y' = t^2 + 2 + t\cos t$$

Solution

$r^4 - r^3 + 2r^2 + 4r = 0$

$$\implies r(r^3 - r^2 + 2r + 4) = 0$$
$$\implies r(r+1)(r^2 - 2r + 4) = 0$$

$\therefore r = 0,\ r = -1,\ r = 1 \pm \sqrt{3}i.\ \therefore y_c = c_1 + c_2 e^{-t} + e^t(c_3 \cos(\sqrt{3}t) + c_4 \sin(\sqrt{3}t))$.

The ansatz is then

$$y = t(At^2 + Bt + C) + (Dt + E)\cos t + (Ft + G)\sin t$$

(The leading t handles the fact that the constant is already in y_c corresponding to $r = 0$.)

4.4 Variation of Parameters for nth Order ODEs

The method of variation of parameters is the same except we need to introduce additional constraints.

Example 14

$y''' - 2y'' - y' + 2y = 2e^{4t}$.

Solution

Characteristic polynomial is $r^3 - 2r^2 - r + 2 = 0$.

$$\implies (r-1)(r^2 - r - 2) = 0 \implies (r-1)(r-2)(r+1) = 0$$

$\therefore r = -1,\ r = 1,\ r = 2$. Thus $y_1 = e^{-t},\ y_2 = e^t,\ y_3 = e^{2t}$. Now the particular solution is of the form

$$y_p = u_1 e^{-t} + u_2 e^t + u_3 e^{2t}$$

where u_1, u_2 and u_3 are functions of t to be determined.

$$y_p' = -u_1 e^{-t} + u_1' e^{-t} + u_2' e^t + u_2 e^t + u_3' e^{2t} + 2u_3 e^{2t}$$

Set $u_1' e^{-t} + u_2' e^t + u_3' e^{2t} = 0$ as arbitrary constraint $(\star)$. Then $y_p' = -u_1 e^{-t} + u_2 e^t + 2u_3 e^{2t}$.

$$y_p'' = u_1 e^{-t} + u_2 e^t - u_1' e^{-t} + u_2' e^t + 2u_3' e^{2t} + 4u_3 e^{2t}$$

Set $u_2' e^t - u_1' e^{-t} + 2u_3' e^{2t} = 0$ as another constraint $(\star\star)$. Then $y_p'' = u_1 e^{-t} + u_2 e^t + 4u_3 e^{2t}$.

$$y_p''' = u_2 e^t + u_1' e^{-t} - u_1 e^{-t} + u_2' e^t + 4u_3' e^{2t} + 8u_3 e^{2t}$$

$$\begin{aligned} y_p''' - 2y_p'' - y_p' + 2y_p &= u_2 e^t + u_1' e^{-t} - u_1 e^{-t} + u_2' e^t + 4u_3' e^{2t} + 8u_3 e^{2t} \\ &\quad - 2u_1 e^{-t} - 2u_2 e^t - 8u_3 e^{2t} + u_1 e^{-t} - u_2 e^t - 2u_3 e^{2t} \\ &\quad + 2u_1 e^{-t} + 2u_2 e^t + 2u_3 e^{2t} \\ &= u_1' e^{-t} + u_2' e^t + 4u_3' e^{2t} = 2e^{4t} \end{aligned}$$

We have three equations:

$$u_1' e^{-t} + u_2' e^t + u_3' e^{2t} = 0 \qquad (1)$$

$$-u_1' e^{-t} + u_2' e^t + 2u_3' e^{2t} = 0 \qquad (2)$$

$$u_1' e^{-t} + u_2' e^t + 4u_3' e^{2t} = 2e^{4t} \qquad (3)$$

5 Series Solutions of Second-Order Linear ODEs

5.1 Review of Power Series

Definition 5.1 (Power Series). A power series centred at x_0 is an infinite series as below. A power series is a polynomial representation of a function,

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{5.1}$$

The power series is said to converge at a point x if

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n(x - x_0)^n \tag{5.2}$$

(the partial sum) exists for that x. A power series may converge for all x, or only some values of x.

The power series converges absolutely at a point x if the associated power series

$$\sum_{n=0}^{\infty} |a_n(x - x_0)^n| = \sum_{n=0}^{\infty} |a_n||x - x_0|^n \tag{5.3}$$

converges. If a power series converges absolutely, it also converges, i.e. absolute convergence $\implies$ convergence.

To test if a power series converges, use the ratio test where if $a_n \neq 0$ and we fix x then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n(x - x_0)^n} \right| = (x - x_0) \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \tag{5.4}$$

Define

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \tag{5.5}$$

- If $L < 1$, the power series converges absolutely.
- If $L = 0$, the test is inconclusive.

- If $L > 1$, the power series does not converge, i.e. it diverges.

If the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges at $x = x_1$, it converges absolutely for $|x - x_0| < |x_1 - x_0|$. If it diverges at $x = x_1$, it diverges for $|x - x_0| > |x_1 - x_0|$.

Definition 5.2 (Radius of Convergence)**.** For each power series, there is a positive number ρ called the radius of convergence such that $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges absolutely for

$$|x - x_0| < \rho \tag{5.6}$$

and diverges for

$$|x - x_0| > \rho \tag{5.7}$$

The interval $|x - x_0| < \rho$ is called the *interval of convergence.*

The series may converge or diverge when $|x - x_0| = \rho$. Some power series converge $\forall x$, thus we say ρ is infinite and the interval of convergence is all of $\mathbb{R}$. Some series only converge at x_0, for that we say $\rho = 0$ and the series has no interval of convergence.

Suppose $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges to $f(x)$ and $\sum_{n=0}^{\infty} b_n(x - x_0)^n$ converges to $g(x)$ for $|x - x_0| < \rho,\ \rho > 0$. Then the two power series can be added and subtracted termwise,

$$f(x) \pm g(x) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x - x_0)^n \tag{5.8}$$

The series converges at least for $|x - x_0| < \rho$. The two series can also be multiplied,

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n(x - x_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(x - x_0)^n\right) = \sum_{n=0}^{\infty} c_n(x - x_0)^n \tag{5.9}$$

where $c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$. The resulting series converges at least for $|x - x_0| < \rho$. If $b_0 \neq 0$, then $g(x_0) \neq 0$ and the series for $f(x)$ can be divided by the series for $g(x)$,

$$\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} d_n(x - x_0)^n \quad \text{(another power series)} \tag{5.10}$$

The coefficients for d_n can be obtained by equating coefficients in

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \left(\sum_{n=0}^{\infty} d_n(x-x_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-x_0)^n\right) \tag{5.11}$$

The function f is also continuous and has derivatives for all orders for $|x-x_0| < \rho$. The derivatives are computed by differentiating the series termwise, i.e.

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \tag{5.12}$$

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1} = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots \tag{5.13}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} = 2a_2 + 6a_3(x-x_0) + \cdots \tag{5.14}$$

Each of these series converges absolutely for $|x-x_0| < \rho$. If $a_n = f^{(n)}(x_0)/n!$, the series is called the *Taylor series* for f about $x = x_0$.

If $\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n$ for each x in some open interval with centre x_0 then $a_n = b_n$ for $n = 0, 1, 2, 3, \ldots$. Also if $\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$ for each x then $a_0 = a_1 = \cdots = a_n = \cdots = 0$.

Definition 5.3 (Analytic Function)**.** A function f is said to be analytic at $x = x_0$ if it has a Taylor series expansion about $x = x_0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \tag{5.15}$$

with a radius of convergence $\rho > 0$.

Most familiar functions are analytic except at certain easily recognised points. $\sin x$, $\cos x$, e^x are analytic everywhere. $1/x$ is analytic everywhere except $x = 0$. If f and g are analytic at x_0 then $f \pm g$, fg and f/g (provided $g(x_0) \neq 0$) is analytic at x_0.

We can also shift the index of summation in an infinite series since it is a dummy variable.

Example 15

$$\sum_{n=0}^{\infty} a_n x^n = a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_{n+2} x^{n+2}$$

Example 16

$$\sum_{n=2}^{\infty}(n+2)(n+1)a_n(x-x_0)^{n-2} = 12a_2+20a_3(x-x_0)+\cdots = \sum_{n=0}^{\infty}(n+4)(n+3)a_{n+2}(x-x_0)^n$$

5.2 Series Solutions Near an Ordinary Point

We saw how to solve second order ODEs with constant coefficients, but we can use power series to solve ODEs with coefficients that are functions of the independent variable,

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{5.16}$$

For now we assume P, Q and R are polynomials with no common factors (if they have common factors, divide it out). If we want to solve an ODE in the form above in the neighbourhood of a point x_0, the solution in an interval x_0 depends on the behaviour of P in that interval.

Definition 5.4 (Ordinary and Singular Points)**.** A point x_0 such that $P(x_0) \neq 0$ is an *ordinary point.* A point x_0 such that $P(x_0) = 0$ is a *singular point.*

If x_0 is ordinary and P is continuous, there is an open interval I where $P(x)$ is never zero. In that interval divide by $P(x)$ to get

$$y'' + p(x)y' + q(x)y = 0 \tag{5.17}$$

Since p and q are continuous functions, there is a unique solution to the ODE above that satisfies $y(x_0) = y_0$ and $y'(x_0) = y_0'$ according to the existence and uniqueness theorem. At a singular point, at least one of p and q becomes unbounded as $x \to x_0$, so the existence and uniqueness theorem does not apply.

To solve the ODE above in the neighbourhood of an ordinary point x_0, we look for a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + \cdots \tag{5.18}$$

and assume the series converges in the interval $|x - x_0| < \rho$ for some $\rho > 0$. We substitute the power series into the ODE and try and find a recurrence relation for the coefficients a_n which then gives us the terms of the power series.

Example 17

Find a power series solution to $y'' - xy' - y = 0$, $x_0 = 0$ and find the first 4 terms in each of the two solutions y_1 and y_2 unless the series terminates sooner.

Solution

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x-0)^n = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots \\
y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + \cdots \\
y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + \cdots
\end{aligned}
$$

$$
\therefore y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0
$$

$$
\implies \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0
$$

We shift the index so they all start at $n = 0$,

$$
\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n
$$

$$
\sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n
$$

$$
\therefore \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} x^n - n a_n x^n - a_n x^n] = 0
$$

$$
= \sum_{n=0}^{\infty} [x^n((n+2)(n+1) a_{n+2} - (n+1) a_n)] = 0
$$

$$
\implies (n+2)(n+1) a_{n+2} - (n+1) a_n = 0 \implies a_{n+2} = \frac{a_n}{n+2} \quad (\text{for } n \geq 0)
$$

At $n = 0$, $a_2 = a_0/2$. At $n = 1$, $a_3 = a_1/3$. At $n = 2$, $a_4 = a_2/4 = a_0/8$. At $n = 3$, $a_5 = a_3/5 = a_1/15$. At $n = 4$, $a_6 = a_4/6 = a_0/48$. At $n = 5$, $a_7 = a_5/7 = a_1/105$. Thus

$$
\begin{aligned}
a_{2k} &= \frac{a_0}{(2k)!!} = \frac{a_0}{2 \cdot 4 \cdot 6 \cdots 2k} \\
a_{2k+1} &= \frac{a_1}{3 \cdot 5 \cdot 7 \cdots (2k+1)}
\end{aligned}
$$

$$
\therefore y_1 = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{1 \cdot 2 \cdot 4 \cdots (2k)}, \quad y_2 = a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{1 \cdot 3 \cdot 5 \cdots (2k+1)}
$$

Example 18

Find a power series solution for $y'' + xy' + 2y = 0$, $x_0 = 0$.

Solution

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

$$
y'' + xy' + 2y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0
$$

$$
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n
$$

$$
\sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n
$$

$$
\therefore \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + na_n + 2a_n]x^n = 0
$$

$$
\implies (n+2)(n+1)a_{n+2} + (n+2)a_n = 0 \implies a_{n+2} = -\frac{a_n}{n+1}, \; n \geq 0
$$

$$
\begin{aligned}
a_2 &= -a_0, \qquad a_3 = -\frac{a_1}{2} \\
a_4 &= -\frac{a_2}{3} = \frac{(-1)^2 a_0}{3} \\
a_5 &= -\frac{a_3}{4} = \frac{(-1)^2 a_1}{8} \\
a_6 &= -\frac{a_4}{5} = \frac{(-1)^3 a_0}{15}
\end{aligned}
$$

$$
a_{2k} = \frac{(-1)^k a_0}{1 \cdot 3 \cdots (2k-1)}, \qquad a_{2k+1} = \frac{(-1)^k a_1}{2 \cdot 4 \cdots (2k)}
$$

$$
\therefore y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{1 \cdot 3 \cdot 5 \cdots (2k-1)} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2 \cdot 4 \cdot 6 \cdots (2k)}
$$

For a_3 onwards we need to differentiate $\phi''(x_0)$. This is okay since $P(x_0) \neq 0$ so all derivatives of P and q are exist. Thus we can assume p and q are analytic functions at x_0. From this we can generalise definitions of ordinary and singular points.

Definition 5.5 (Ordinary Point (General)). If $p = Q/P$ and $q = R/P$ are analytic at some point x_0, then x_0 is called an *ordinary point*. Otherwise, x_0 is called a *singular point*.

The following result (not proved) tells us how to determine the radius of convergence of a solution around x_0.

Theorem 5.1. If x_0 is an ordinary point of the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$, that is if $p = Q/P$ and $q = R/P$ are analytic at x_0, then the general solution of the ODE is

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 y_1(x) + a_1 y_2(x) \tag{5.19}$$

where a_0 and a_1 are arbitrary and y_1, y_2 are linearly independent series solutions that are analytic at x_0. Further, the radius of convergence for each of the series solutions y_1, y_2 is at least as large as the minimum of the radii of convergence of the series for p and q.

Now assume we have

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{5.20}$$

where P, Q, R are any functions of the independent variable x with $P(x_0) \neq 0$. Assume we have a power series solution of the form

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{5.21}$$

Assume this converges in the interval $|x - x_0| < \rho$ for some $\rho > 0$. Differentiate ϕ m times and set $x = x_0$ to obtain

$$\phi^{(m)}(x_0) = m! a_m \implies a_m = \frac{\phi^{(m)}(x_0)}{m!} \tag{5.22}$$

Thus to compute a_n, we need to determine $\phi^{(n)}$ for $n = 0, 1, 2, \ldots$. Assume we have the

IC $\phi(x_0) = a_0 = y_0$ and $\phi'(x_0) = a_1 = y_0'$. For the other a_n use

$$\phi''(x) = -\frac{Q(x)\phi'(x) + R(x)\phi(x)}{P(x)} \tag{5.23}$$

Thus $2!a_2 = \phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0)$ where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$.

5.3 Euler Equations

Definition 5.6 (Euler Equation)**.** An Euler equation is an ODE of the form

$$L[y] = x^2y'' + \alpha xy' + \beta y = 0, \qquad \alpha, \beta \in \mathbb{R} \tag{5.24}$$

Here $P(x) = x^2,\ Q(x) = \alpha x,\ R(x) = \beta$. The only singular point (when $\beta \neq 0$) is $x = 0$.

We want to extend the method of solutions for ordinary points to solve Euler equations. These occur when our singularities are weak.

Definition 5.7 (Regular Singular Point)**.** A point x_0 is a regular singular point if

$$\lim_{x\to x_0}(x - x_0)\frac{Q(x)}{P(x)} \text{ and } \lim_{x\to x_0}(x - x_0)^2\frac{R(x)}{P(x)} \text{ are finite} \tag{5.25}$$

This means the singularity in Q/P is no worse than $(x - x_0)^{-1}$ and the singularity in R/P is no worse than $(x - x_0)^{-2}$. Such a point is called a *regular singular point.*

For more general ODEs of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{5.26}$$

x_0 is a regular singular point if it is a singular point ($P(x_0) = 0$) and if both

$$(x - x_0)\frac{Q(x)}{P(x)} \text{ and } (x - x_0)^2\frac{R(x)}{P(x)} \tag{5.27}$$

are analytic at $x = x_0$, i.e. they have convergent Taylor series at $x = x_0$.

We can test if $x = 0$ is a regular singular point for the Euler equation

$$\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} = \lim_{x \to 0} x \cdot \frac{\alpha x}{x^2} = \lim_{x \to 0} \alpha = \alpha \quad \text{(finite)} \tag{5.28}$$

$$\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} = \lim_{x \to 0} x^2 \cdot \frac{\beta}{x^2} = \beta \quad \text{(finite)} \tag{5.29}$$

Thus $x = 0$ is a regular singular point.

Now for any interval not including the origin, the existence and uniqueness theorem says that the Euler equation has two linearly independent solutions

$$y = c_1 y_1 + c_2 y_2 \tag{5.30}$$

As an ansatz let $y = x^r$. Then

$$y = x^r \tag{5.31}$$

$$y' = r x^{r-1} \tag{5.32}$$

$$y'' = r(r-1) x^{r-2} \tag{5.33}$$

$$\implies x^2 y'' + \alpha x y' + \beta y = 0 \tag{5.34}$$

$$\implies r(r-1)x^r + \alpha r x^r + \beta x^r = 0 \tag{5.35}$$

$$\implies x^r (r(r-1) + \alpha r + \beta) = 0 \tag{5.36}$$

$$\implies x^r (r^2 + (\alpha - 1) r + \beta) = 0 \tag{5.37}$$

Since $x^r \neq 0$, we get a characteristic polynomial

$$r^2 + (\alpha - 1) r + \beta = 0 \tag{5.38}$$

5.3.0.1 Case 1: Two real roots r_1 and r_2.

Our solution is then

$$y = c_1 x^{r_1} + c_2 x^{r_2} \tag{5.39}$$

We can check for linear independence via the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} x^{r_1} & x^{r_2} \\ r_1 x^{r_1 - 1} & r_2 x^{r_2 - 1} \end{vmatrix} = r_2 x^{r_1 + r_2 - 1} - r_1 x^{r_1 + r_2 - 1} \tag{5.40}$$

$$= x^{r_1 + r_2 - 1}(r_2 - r_1) \tag{5.41}$$

Since $r_1 \neq r_2$, $W(y_1, y_2) \neq 0$ so we have linear independence. If r is not a rational number write $x^r = e^{r \ln x} \implies y = c_1 e^{r_1 \ln x} + c_2 e^{r_2 \ln x}$.

5.3.0.2 Case 2: Equal roots ($r_1 = r_2$).

We have one solution $y_1 = x^{r_1}$. For the second one we use reduction of order. Let

$$y_2 = u y_1 \tag{5.42}$$

$$\implies y_2' = u' y_1 + u y_1' \tag{5.43}$$

$$\implies y_2'' = u'' y_1 + u' y_1' + u' y_1' + u y_1'' = u'' y_1 + 2u' y_1' + u y_1'' \tag{5.44}$$

Thus,

$$x^2 y_2'' + \alpha x y_2' + \beta y_2 = x^2 (u'' y_1 + 2u' y_1' + u y_1'') + \alpha x (u' y_1 + u y_1') + \beta u y_1 \tag{5.45}$$

$$= x^2 y_1 u'' + (\alpha x y_1 + 2x^2 y_1') u' + (x^2 y_1'' + \alpha x y_1' + \beta y_1) u = 0 \tag{5.46}$$

Since y_1 solves the ODE, the last term vanishes,

$$\implies x^2 y_1 u'' + (\alpha x y_1 + 2x^2 y_1')u' = 0 \tag{5.47}$$

$$\implies x^{r_1+2}u'' + (\alpha x^{r_1+1} + 2r_1 x^2 x^{r_1-1})u' = 0 \tag{5.48}$$

$$\implies x^{r_1+1}(xu'' + (\alpha + 2r_1)u') = 0 \tag{5.49}$$

Define $w = u'$. Then

$$xw' + (\alpha + 2r_1)w = 0 \tag{5.50}$$

$$\implies \frac{w'}{w} = -\frac{\alpha + 2r_1}{x} \tag{5.51}$$

$$\implies \ln|w| = -(\alpha + 2r_1)\ln x + c_3 \tag{5.52}$$

$$\implies u' = w = \pm e^{c_3} x^{-(\alpha+2r_1)} = c_2 x^{-(\alpha+2r_1)} \tag{5.53}$$

Thus since $r_1 = (1-\alpha)/2$, we have $\alpha + 2r_1 = 1$. Thus

$$u' = c_2 x^{-1} \tag{5.54}$$

$$u = c_2 \ln x + c_1 \implies y = uy_1 = c_1 x^{r_1} + c_2 x^{r_1} \ln x$$

Thus $y_2 = x^{r_1} \ln x$. Checking linear independence,

$$W(y_1, y_2) = \begin{vmatrix} x^{r_1} & x^{r_1}\ln x \\ r_1 x^{r_1-1} & r_1 x^{r_1-1}\ln x + x^{r_1-1} \end{vmatrix}$$

$$= r_1 x^{2r_1-1}\ln x + x^{2r_1-1} - r_1 x^{2r_1-1}\ln x = x^{2r_1-1} \neq 0$$

Thus the general solution is

$$y = c_1 x^{r_1} + c_2 x^{r_1} \ln x \tag{5.55}$$

5.3.0.3 Case 3: Complex roots.

Assume the complex conjugate roots are

$$r = \lambda \pm i\mu \text{ where } \mu \neq 0 \tag{5.56}$$

Then

$$x^r = x^{\lambda+i\mu} = x^\lambda e^{i\mu \ln x} = x^\lambda[\cos(\mu \ln x) + i\sin(\mu \ln x)] \tag{5.57}$$

$$\bar{x^r} = x^{\lambda-i\mu} = x^\lambda e^{-i\mu \ln x} = x^\lambda[\cos(\mu \ln x) - i\sin(\mu \ln x)] \tag{5.58}$$

Then by a similar trick as we did with complex roots of the characteristic polynomial we get

$$y_1 = x^\lambda \cos(\mu \ln x), \qquad y_2 = x^\lambda \sin(\mu \ln x) \tag{5.59}$$

Thus the general solution is

$$y = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x) \tag{5.60}$$

Confirming linear independence,

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} x^\lambda \cos(\mu \ln x) & x^\lambda \sin(\mu \ln x) \\ \lambda x^{\lambda-1}\cos(\mu \ln x) - \frac{\mu}{x}x^\lambda \sin(\mu \ln x) & \lambda x^{\lambda-1}\sin(\mu \ln x) + \frac{\mu}{x}x^\lambda \cos(\mu \ln x) \end{vmatrix} \\
&= \lambda x^{2\lambda-1}\sin(\mu \ln x)\cos(\mu \ln x) + \mu x^{2\lambda-1}\cos^2(\mu \ln x) \\
&\quad - \lambda x^{2\lambda-1}\sin(\mu \ln x)\cos(\mu \ln x) + \mu x^{2\lambda-1}\sin^2(\mu \ln x) \\
&= \mu x^{2\lambda-1} \neq 0
\end{aligned}$$

6 Systems of First-Order Linear Equations

Many real world systems involve separate but interconnected components that depend on each other such as current and voltage in a circuit, predator and prey populations etc. Thus several quantities tend to interact with each other but they depend on the same independent variable, typically time. To model systems like these we use a system of ODEs.

We can also derive a system of ODEs from higher order ODEs. Consider the damped-free spring mass system

$$mx'' + \gamma x' + kx = 0 \tag{6.1}$$

Let $u_1 = x$ and $u_2 = x'$. Then $u_1' = x' = u_2$ and $u_2' = x''$. Thus,

$$u_1' = u_2 \tag{6.2}$$

$$mu_2' = -ku_1 - \gamma u_2 \tag{6.3}$$

To transform an arbitrary nth order equation into a system of n first-order ODEs

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)}) \tag{6.4}$$

Start by defining

$$x_1 = y,\ x_2 = y',\ \ldots,\ x_n = y^{(n-1)} \tag{6.5}$$

Then it follows that

$$x_1' = x_2 \tag{6.6}$$

$$x_2' = x_3 \tag{6.7}$$

$$\vdots \tag{6.8}$$

$$x_{n-1}' = x_n \tag{6.9}$$

Then,

$$x_1' = F_1(t, x_1, \dots, x_n) \tag{6.10}$$

$$x_2' = F_2(t, x_1, \dots, x_n) \tag{6.11}$$

$$\vdots \tag{6.12}$$

$$x_n' = F_n(t, x_1, \dots, x_n) \tag{6.13}$$

A solution of such a system on the open interval $I = (\alpha, \beta)$ consists of n functions

$$x_1 = \phi_1,\ x_2 = \phi_2,\ \dots,\ x_n = \phi_n \tag{6.14}$$

where each function is differentiable at all points on the interval and the system is satisfied at all points in the interval. In addition we may specify n initial conditions to get an IVP,

$$x_1(t_0) = x_1^0,\ x_2(t_0) = x_2^0,\ \dots,\ x_n(t_0) = x_n^0 \tag{6.15}$$

A solution to such a system is a set of parametric equations in n-dimensional space. As time changes we say the trajectory of the system changes which is specified by a curve in n-dimensional space. The initial conditions specify the starting point of the system.

There is an existence and uniqueness theorem for such a system.

Theorem 6.1 (Existence and Uniqueness)**.** Let each of the n functions $F_1, \dots, F_n$ and the n^2 first partial derivatives $\partial F_1/\partial x_1, \dots, \partial F_1/\partial x_n, \dots, \partial F_n/\partial x_1, \dots, \partial F_n/\partial x_n$ be continuous in a region R of $t, x_1, x_2, \dots, x_n$ space defined by $\alpha < t < \beta,\ \alpha_1 < x_1 < \beta_1,\ \dots,\ \alpha_n < x_n < \beta_n$ and let the point $(t_0, x_1^0, x_2^0, \dots, x_n^0)$ be in R. Then there is an interval $|t - t_0| < h$ in which there exists a unique solution $x_1 = \phi_1(t), \dots, x_n = \phi_n(t)$ of the system of differential equations that also satisfies the initial condition.

Remark. Nothing is said about $\partial F_i/\partial t$.

If each of the functions $F_1, \dots, F_n$ is a linear function of the dependent variables $x_1, \dots, x_n$ then the system of differential equations is said to be *linear*; otherwise it is non-linear.

The most general system of n first-order linear differential equations has the form

$$x_1' = p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \tag{6.16}$$

$$x_2' = p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t) \tag{6.17}$$

$$\vdots \tag{6.18}$$

$$x_n' = p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \tag{6.19}$$

We can write this as a matrix equation

$$\mathbf{x}' = A\mathbf{x} + \mathbf{G} \tag{6.20}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \; A = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix}, \; \mathbf{G} = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix} \tag{6.21}$$

If $\mathbf{G} = 0$, the system is said to be *homogeneous*. We can state a stronger version of the existence and uniqueness theorem for linear systems.

Theorem 6.2. If the functions $p_{11}, p_{12}, \ldots, p_{nn},\; g_1, \ldots, g_n$ are continuous on an open interval $I = (\alpha, \beta)$, then there exists a unique solution $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ of the system that also satisfies the initial conditions where $t_0 \in I$ and $x_1^0, \ldots, x_n^0$ are any prescribed. The solution also exists throughout the interval I.

Before we discuss solutions, we need to understand eigenvectors and eigenvalues. If A is a matrix and $\mathbf{v}$ is a non-zero vector such that

$$A\mathbf{v} = \lambda\mathbf{v} \tag{6.22}$$

then $\mathbf{v}$ is said to be an eigenvector and λ is the corresponding eigenvalue of the eigenvector. Subtract to get

$$A\mathbf{v} - \lambda I\mathbf{v} = 0 \implies (A - \lambda I)\mathbf{v} = 0 \tag{6.23}$$

The trivial solution for $\mathbf{v}$ is $\mathbf{v} = 0$, but since $\mathbf{v}$ is an eigenvector $\mathbf{v} \neq 0$. Thus we need to ensure $A - \lambda I$ is not invertible which is true only if its determinant is 0. Thus we can compute the eigenvalues of A by solving the equation

$$\det(A - \lambda I) = 0 \tag{6.24}$$

6.1 Real Eigenvalues of the Coefficient Matrix

Assume our system is of the form

$$x_1' = ax_1 + bx_2 \tag{6.25}$$

$$x_2' = cx_1 + dx_2 \tag{6.26}$$

This can be written as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6.27}$$

If the eigenvalues of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are real, say λ_1, λ_2, then an associated eigenvector we denote by $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ satisfies

$$A\boldsymbol{\xi}_1 = \lambda\boldsymbol{\xi}_1 \tag{6.28}$$

Let our ansatz be

$$\mathbf{x}(t) = e^{\lambda t}\boldsymbol{\xi} \tag{6.29}$$

Since $\boldsymbol{\xi}$ will be a function of t,

$$A\mathbf{x}(t) = Ae^{\lambda t}\boldsymbol{\xi} = A\boldsymbol{\xi}e^{\lambda t} = \lambda\boldsymbol{\xi}e^{\lambda t} \tag{6.30}$$

$$\mathbf{x}'(t) = \lambda e^{\lambda t}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}e^{\lambda t} = A\boldsymbol{\xi}e^{\lambda t} = A\mathbf{x}(t) \tag{6.31}$$

Thus $\mathbf{x}(t) = e^{\lambda t}\boldsymbol{\xi}$ is a solution. For our two eigenvalues we write this as

$$\mathbf{x}_1 = e^{\lambda_1 t}\boldsymbol{\xi}_1 \tag{6.32}$$

$$\mathbf{x}_2 = e^{\lambda_2 t}\boldsymbol{\xi}_2 \tag{6.33}$$

By the principle of superposition, the general solution is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\boldsymbol{\xi}_1 + c_2 e^{\lambda_2 t}\boldsymbol{\xi}_2 \tag{6.34}$$

Example 19

$x_1' = 3x_1 - 2x_2,\ x_2' = 2x_1 - 2x_2.$

Solution

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} = 0$$

$$\implies (3-\lambda)(-2-\lambda) + 4 = 0$$

$$\implies -6 - 3\lambda + 2\lambda + \lambda^2 + 4 = 0$$

$$\implies \lambda^2 - \lambda - 2 = 0 \implies (\lambda - 2)(\lambda + 1) = 0 \implies \lambda = 2,\ \lambda = -1$$

For $\lambda = 2$,

$$\begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies v_1 - 2v_2 = 0 \implies v_1 = 2v_2 \implies \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda = -1$,

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 4v_1 = 2v_2 \implies 2v_1 = v_2 \implies \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore \mathbf{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

6.2 Complex Eigenvalues of the Coefficient Matrix

Assume our system is

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{6.35}$$

The eigenvalues ρ of A are given by

$$\det(A-\rho I) = \det\begin{pmatrix} a-\rho & b \\ c & d-\rho \end{pmatrix} = (a-\rho)(d-\rho) - bc = 0 \tag{6.36}$$

Expanding,

$$\rho^2 - (a+d)\rho + (ad-bc) = 0 \tag{6.37}$$

Assume we have a complex eigenvalue given by

$$\rho_1 = \lambda + i\mu \tag{6.38}$$

Then its complex conjugate must be the other eigenvalue since the coefficients of the matrix are still real. Thus,

$$\rho_1 = \lambda + i\mu,\ \rho_2 = \lambda - i\mu \tag{6.39}$$

Define our eigenvalue polynomial as

$$P(\rho) = \rho^2 - (a+d)\rho + (ad-bc) \tag{6.40}$$

Then $P(\bar{\rho}) = \bar{\rho}^2 - (a+d)\bar{\rho} + ad - bc = \overline{P(\rho)}$. Let us denote our eigenvector of ρ as $\boldsymbol{\xi}$. Then

$$(A-\rho I)\boldsymbol{\xi} = 0 \implies \overline{(A-\rho I)\boldsymbol{\xi}} = 0 \tag{6.41}$$

Thus,

$$(A-\bar{\rho}I)\bar{\boldsymbol{\xi}} = 0 \tag{6.42}$$

Thus, the two solutions are

$$\mathbf{x}_1 = \boldsymbol{\xi}e^{\rho t},\ \mathbf{x}_2 = \bar{\boldsymbol{\xi}}e^{\bar{\rho}t} \tag{6.43}$$

Since $\rho = \lambda + i\mu$ we can use Euler's formula and set $\boldsymbol{\xi} = \boldsymbol{\sigma} + i\boldsymbol{\tau}$ to get

$$\begin{aligned}
\mathbf{x}_1 &= (\boldsymbol{\sigma} + i\boldsymbol{\tau})e^{\lambda t + i\mu t} && (6.44)\\
&= e^{\lambda t}(\boldsymbol{\sigma} + i\boldsymbol{\tau})e^{i\mu t} && (6.45)\\
&= e^{\lambda t}(\boldsymbol{\sigma} + i\boldsymbol{\tau})(\cos\mu t + i\sin\mu t) && (6.46)\\
&= e^{\lambda t}(\boldsymbol{\sigma}\cos\mu t - \boldsymbol{\tau}\sin\mu t) + ie^{\lambda t}(\boldsymbol{\tau}\cos\mu t + \boldsymbol{\sigma}\sin\mu t) && (6.47)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{x}_2 &= (\boldsymbol{\sigma} - i\boldsymbol{\tau})e^{\lambda t - i\mu t} && (6.48)\\
&= e^{\lambda t}(\boldsymbol{\sigma} - i\boldsymbol{\tau})(\cos\mu t - i\sin\mu t) && (6.49)\\
&= e^{\lambda t}(\boldsymbol{\sigma}\cos\mu t - \boldsymbol{\tau}\sin\mu t) - ie^{\lambda t}(\boldsymbol{\tau}\cos\mu t + \boldsymbol{\sigma}\sin\mu t) && (6.50)
\end{aligned}$$

To obtain real-valued solutions define

$$\begin{aligned}
\mathbf{u}(t) &= \frac{\mathbf{x}_1 + \mathbf{x}_2}{2} = \mathrm{Re}(\mathbf{x}_1) = e^{\lambda t}(\boldsymbol{\sigma}\cos\mu t - \boldsymbol{\tau}\sin\mu t) && (6.51)\\
\mathbf{v}(t) &= \frac{\mathbf{x}_1 - \mathbf{x}_2}{2i} = \mathrm{Im}(\mathbf{x}_1) = e^{\lambda t}(\boldsymbol{\tau}\cos\mu t + \boldsymbol{\sigma}\sin\mu t) && (6.52)
\end{aligned}$$

The general solution is then

$$\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) \tag{6.53}$$

Thus,

$$\mathbf{x}(t) = c_1e^{\lambda t}(\boldsymbol{\sigma}\cos\mu t - \boldsymbol{\tau}\sin\mu t) + c_2e^{\lambda t}(\boldsymbol{\tau}\cos\mu t + \boldsymbol{\sigma}\sin\mu t) \tag{6.54}$$

Example 20

$\mathbf{x}' = \begin{pmatrix} -1 & -5 \\ 1 & 3 \end{pmatrix}\mathbf{x}$ with $x_1(0) = -1$ and $x_2(0) = 1$.

Solution

$$\det\begin{pmatrix}-1-\lambda & -5\\ 1 & 3-\lambda\end{pmatrix} = (3-\lambda)(-1-\lambda)+5=0$$

$$\implies \lambda^2-2\lambda+2=0 \implies \lambda=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm 2i}{2}=1\pm i$$

For $\rho = 1+i$ ($\lambda=1, \mu=1$),

$$\begin{pmatrix}-1 & -5\\ 1 & 3\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = (1+i)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$$

$$\implies -\xi_1-5\xi_2=(1+i)\xi_1 \implies \xi_1=\frac{-5\xi_2}{2+i}=-(2-i)\xi_2$$

Let $\xi_2=1$, so $\xi_1=-(2-i)=-2+i$. Thus

$$\boldsymbol{\xi}=\begin{pmatrix}-2+i\\ 1\end{pmatrix}=\begin{pmatrix}-2\\ 1\end{pmatrix}+i\begin{pmatrix}1\\ 0\end{pmatrix}$$

$\boldsymbol{\sigma}=(-2,1)^T$, $\boldsymbol{\tau}=(1,0)^T$. Thus

$$\mathbf{x}_1=e^t(\cos t+i\sin t)\begin{pmatrix}-2+i\\ 1\end{pmatrix}$$

Taking real and imaginary parts,

$$\mathbf{x}(t)=c_1\begin{pmatrix}-e^t\sin t-2e^t\cos t\\ e^t\cos t\end{pmatrix}+c_2\begin{pmatrix}e^t\cos t-2e^t\sin t\\ e^t\sin t\end{pmatrix}$$

At $t=0$,

$$\begin{pmatrix}x_1(0)\\ x_2(0)\end{pmatrix}=\begin{pmatrix}-2c_1+c_2\\ c_1\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix}$$

Thus $c_1=1$ and $-2+c_2=-1 \implies c_2=1$. So

$$\mathbf{x}(t)=\begin{pmatrix}-e^t\cos t-3e^t\sin t\\ e^t\cos t+e^t\sin t\end{pmatrix}$$

6.3 Repeated Eigenvalues of the Coefficient Matrix

Assume our system is

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{6.55}$$

The eigenvalues ρ of A are given by

$$\det(A - \rho I) = (a - \rho)(d - \rho) - bc = 0 \tag{6.56}$$

Expanding,

$$\rho^2 - (a + d)\rho + (ad - bc) = 0 \tag{6.57}$$

If we have a repeated eigenvalue, then the discriminant is 0,

$$(a + d)^2 - 4(ad - bc) = 0 \tag{6.58}$$

Let the eigenvalue ρ be repeated and its corresponding eigenvector $\boldsymbol{\xi}$ so we have

$$\mathbf{x} = \boldsymbol{\xi} e^{\rho t} \tag{6.59}$$

A second solution of this form exists only under special circumstances, namely when our coefficient matrix is diagonal since that gives two linearly independent eigenvectors.

We look for solutions containing the term $te^{\rho t}$. We know from linear algebra about the concept of a *generalised eigenvector* $\boldsymbol{\eta}$ corresponding to the eigenvalue ρ which is defined through

$$(A - \rho I)\boldsymbol{\eta} = \boldsymbol{\xi} \implies A\boldsymbol{\eta} = \boldsymbol{\xi} + \rho\boldsymbol{\eta} \tag{6.60}$$

We define

$$\mathbf{x}_2(t) = \boldsymbol{\xi} t e^{\rho t} + \boldsymbol{\eta} e^{\rho t} \tag{6.61}$$

Then,

$$A\mathbf{x}_2(t) = te^{\rho t}A\boldsymbol{\xi} + e^{\rho t}A\boldsymbol{\eta} \tag{6.62}$$

$$= te^{\rho t}\rho\boldsymbol{\xi} + e^{\rho t}(\boldsymbol{\xi} + \rho\boldsymbol{\eta}) \tag{6.63}$$

$$= \boldsymbol{\xi}e^{\rho t} + \rho\boldsymbol{\xi}te^{\rho t} + \rho\boldsymbol{\eta}e^{\rho t} \tag{6.64}$$

Then,

$$\mathbf{x}_2'(t) = \boldsymbol{\xi}e^{\rho t} + \rho\boldsymbol{\xi}te^{\rho t} + \rho\boldsymbol{\eta}e^{\rho t} \tag{6.65}$$

Thus, $\mathbf{x}_2' = A\mathbf{x}_2$ so $\mathbf{x}_2$ satisfies the equation. Thus the general solution is given by

$$\mathbf{x}(t) = c_1e^{\rho t}\boldsymbol{\xi} + c_2(\boldsymbol{\xi}te^{\rho t} + \boldsymbol{\eta}e^{\rho t}) \tag{6.66}$$

where ρ is the repeated eigenvalue, $\boldsymbol{\xi}$ is the corresponding eigenvector and $\boldsymbol{\eta}$ is the generalised eigenvector.

Example 21

$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\mathbf{x}$ with $x_1(0) = 3$ and $x_2(0) = 2$.

Solution

$$\det\begin{pmatrix}1-\rho & -4\\ 4 & -7-\rho\end{pmatrix}=0 \implies (1-\rho)(-7-\rho)+16=0 \implies \rho^2+6\rho+9=0 \implies (\rho+3)^2=0$$

So $\rho=-3$. For $\rho=-3$,

$$\begin{pmatrix}1-(-3) & -4\\ 4 & -7-(-3)\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\implies \begin{pmatrix}4 & -4\\ 4 & -4\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=0 \implies \xi_1=\xi_2 \implies \boldsymbol{\xi}=\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\therefore \mathbf{x}_1(t)=e^{-3t}\begin{pmatrix}1\\ 1\end{pmatrix}$$

Now

$$\begin{pmatrix}1-(-3) & -4\\ 4 & -7-(-3)\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\implies \begin{pmatrix}4 & -4\\ 4 & -4\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\implies 4\eta_1-4\eta_2=1 \implies 4\eta_1=1+4\eta_2$$

Let $\eta_2=1$, then $\eta_1=5/4$. So $\boldsymbol{\eta}=\begin{pmatrix}5/4\\ 1\end{pmatrix}$.

$$\therefore \mathbf{x}(t)=c_1e^{-3t}\begin{pmatrix}1\\ 1\end{pmatrix}+c_2\left(te^{-3t}\begin{pmatrix}1\\ 1\end{pmatrix}+e^{-3t}\begin{pmatrix}5/4\\ 1\end{pmatrix}\right)$$

Substitute initial conditions to solve for c_1, c_2.

6.3.1 Drawing Phase Portraits

The phase portrait of a 2D linear system $\mathbf{x}' = A\mathbf{x}$ is determined entirely by the eigenvalues of A. There are five qualitatively different cases.

6.3.1.1 Case 1: Real, distinct eigenvalues of opposite sign ($\rho_1 < 0 < \rho_2$).

We have a *saddle*. Trajectories come in along the stable eigendirection (corresponding to $\rho_1 < 0$) and leave along the unstable eigendirection (corresponding to $\rho_2 > 0$).

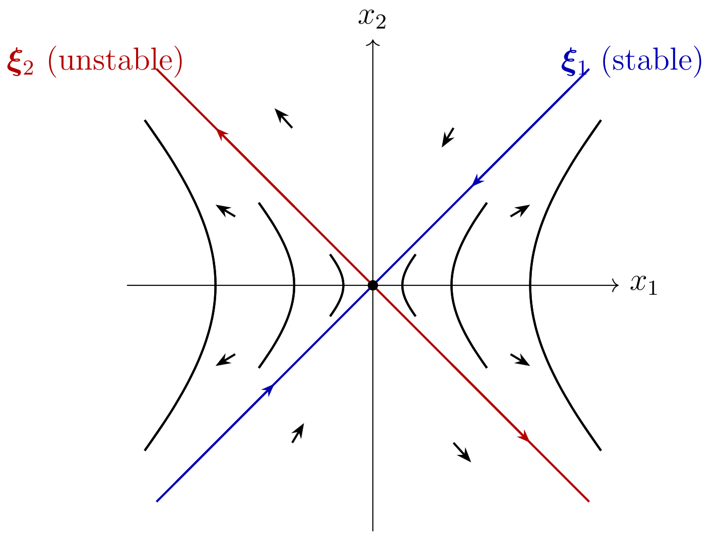


Figure 10: Saddle: $\rho_1 < 0 < \rho_2$.

6.3.1.2 Case 2: Real eigenvalues both negative ($\rho_1 < \rho_2 < 0$).

We have a *stable node* (sink). All trajectories converge to the origin as $t \to \infty$. Near the origin, trajectories are tangent to the slow eigenvector (corresponding to ρ_2, the eigenvalue closer to 0), and far from the origin they are tangent to the fast eigenvector (corresponding to ρ_1).

6.3.1.3 Case 3: Real eigenvalues both positive ($0 < \rho_2 < \rho_1$).

We have an *unstable node* (source). All trajectories move away from the origin as $t \to \infty$. The portrait looks like the stable node with arrows reversed.

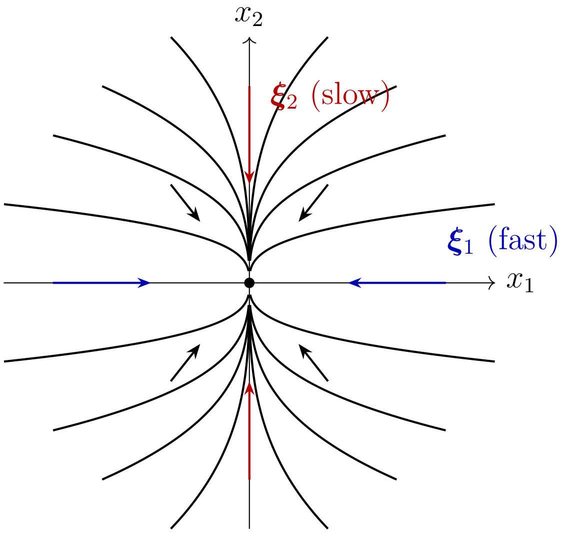


Figure 11: Stable node: $\rho_1 < \rho_2 < 0$.

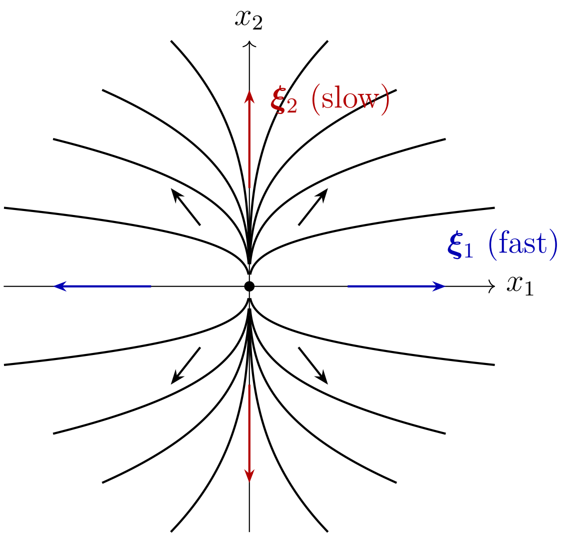


Figure 12: Unstable node: $0 < \rho_2 < \rho_1$.

6.3.1.4 Case 4: Complex conjugate eigenvalues with zero real part ($\rho = \pm i\mu$).

We have a *centre*. Trajectories are closed ellipses around the origin; the origin is stable but not asymptotically stable.

6.3.1.5 Case 5: Complex conjugate eigenvalues with non-zero real part ($\rho = \lambda \pm i\mu$).

We have a *spiral*. If $\lambda < 0$, trajectories spiral inward (stable spiral / stable focus). If $\lambda > 0$, trajectories spiral outward (unstable spiral / unstable focus).

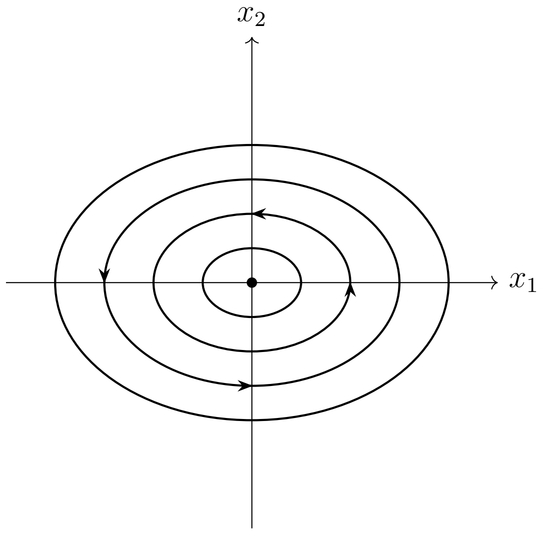
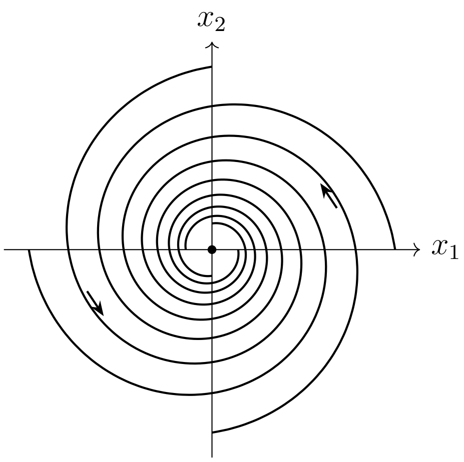
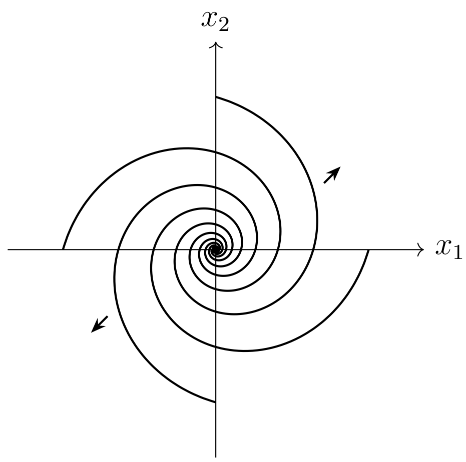


Figure 13: Centre: $\rho = \pm i\mu$, stable but not asymptotically stable.



Stable spiral: $\lambda < 0$.



Unstable spiral: $\lambda > 0$.

Figure 14: Spiral: $\rho = \lambda \pm i\mu$ with $\lambda \neq 0$.

6.3.1.6 Case 6: Degenerate node (repeated eigenvalue).

When A has a repeated eigenvalue ρ with only one linearly independent eigenvector $\boldsymbol{\xi}$, trajectories approach (or leave, depending on sign of ρ) the origin tangent to $\boldsymbol{\xi}$, but they turn to align with it. This is called a *degenerate node* or *improper node*.

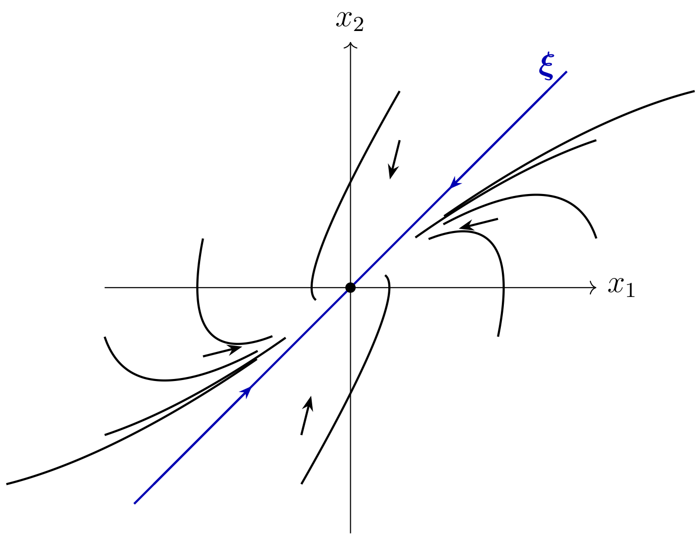


Figure 15: Degenerate stable node: $\rho < 0$ repeated with one eigenvector.

6.3.1.7 Summary.

Eigenvalues	Type	Stability
$\rho_1 < 0 < \rho_2$	Saddle	Unstable
$\rho_1, \rho_2 < 0$ distinct	Stable node	Asymp. stable
$0 < \rho_1, \rho_2$ distinct	Unstable node	Unstable
$\rho = \pm i\mu$	Centre	Stable (not asymp.)
$\rho = \lambda \pm i\mu,\ \lambda < 0$	Stable spiral	Asymp. stable
$\rho = \lambda \pm i\mu,\ \lambda > 0$	Unstable spiral	Unstable
$\rho < 0$ repeated (1 eigenvector)	Degenerate stable node	Asymp. stable
$\rho > 0$ repeated (1 eigenvector)	Degenerate unstable node	Unstable

6.4 Fundamental Matrices

Consider again the two dimensional system

$$\mathbf{x}' = A\mathbf{x} \tag{6.67}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.68}$$

and the initial conditions $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$. The eigenvalues of A determine what the solutions of the system look like.

If A has real distinct eigenvalues ρ_1, ρ_2 with corresponding eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, the general solution is

$$\mathbf{x}(t) = c_1 e^{\rho_1 t}\boldsymbol{\xi}_1 + c_2 e^{\rho_2 t}\boldsymbol{\xi}_2 \tag{6.69}$$

Also, $\mathbf{x}(0) = c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$.

If A has a pair of complex conjugate eigenvalues $\lambda \pm i\mu$ with corresponding eigenvectors $\boldsymbol{\xi} = \boldsymbol{\sigma} + i\boldsymbol{\tau}$, the general solution is

$$\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) \tag{6.70}$$

where $\mathbf{u}(t) = \mathrm{Re}(e^{\rho t}\boldsymbol{\xi})$ and $\mathbf{v}(t) = \mathrm{Im}(e^{\rho t}\boldsymbol{\xi})$. Also, $\mathbf{x}(0) = c_1\boldsymbol{\sigma} + c_2\boldsymbol{\tau} = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$.

If A has a repeated eigenvalue ρ with eigenvector $\boldsymbol{\xi}$ then the general solution is

$$\mathbf{x}(t) = c_1 e^{\rho t}\boldsymbol{\xi} + c_2(te^{\rho t}\boldsymbol{\xi} + e^{\rho t}\boldsymbol{\eta}) \tag{6.71}$$

where $(A - \rho I)\boldsymbol{\eta} = \boldsymbol{\xi}$. Also, $\mathbf{x}(0) = c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta} = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$.

We can encode all this information without worrying about the three different cases by considering a *fundamental matrix.*

Definition 6.1 (Fundamental Matrix)**.** The fundamental matrix $\Phi(t)$ is a matrix whose columns are the vectors that represent the linearly independent solutions above. Since its columns are linearly independent vectors, $\Phi(t)$ is invertible.

We construct $\hat{\Phi}(t)$ such that $\hat{\Phi}(0) = I$. Given our initial conditions, the general solution to the IVP is given by

$$\mathbf{x}(t) = \hat{\Phi}(t)\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \tag{6.72}$$

We define the fundamental matrix by starting with considering what it means to have a matrix as a power. We define

$$\exp(At) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!} \tag{6.73}$$

Also,

$$\frac{d}{dt}[\exp(At)] = \frac{d}{dt}\left[I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}\right] \tag{6.74}$$

$$= \sum_{n=1}^{\infty} \frac{A^n t^{n-1}}{(n-1)!} \tag{6.75}$$

Also,

$$A\exp(At) = A + \sum_{n=1}^{\infty} \frac{A^{n+1} t^n}{n!} = \sum_{n=0}^{\infty} \frac{A^{n+1} t^n}{n!} = \sum_{k=1}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} \tag{6.76}$$

Thus,

$$\frac{d}{dt}(\exp(At)) = A\exp(At) \tag{6.77}$$

Also,

$$\exp(At)|_{t=0} = \exp(0) = I \tag{6.78}$$

Since $\exp(At)$ satisfies the IC and its derivative is continuous, by the existence and uniqueness theorem for linear systems we can say that

$$\hat{\Phi}(t) = \exp(At) \tag{6.79}$$

Example 22

Let $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$. Since A is diagonal, its eigenvalues are a_1 and a_2. Also $A^2 = \begin{pmatrix} a_1^2 & 0 \\ 0 & a_2^2 \end{pmatrix}$, $A^3 = \begin{pmatrix} a_1^3 & 0 \\ 0 & a_2^3 \end{pmatrix}$, $\ldots$, $A^n = \begin{pmatrix} a_1^n & 0 \\ 0 & a_2^n \end{pmatrix}$.

Solution

Thus,

$$\exp(At) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!} = \begin{pmatrix} 1 + \sum_{n=1}^{\infty} \frac{a_1^n t^n}{n!} & 0 \\ 0 & 1 + \sum_{n=1}^{\infty} \frac{a_2^n t^n}{n!} \end{pmatrix}$$

Thus,

$$\exp(At) = \begin{pmatrix} e^{a_1 t} & 0 \\ 0 & e^{a_2 t} \end{pmatrix}$$

A way to compute the fundamental matrix is via Putzer's algorithm.

Theorem 6.3 (Putzer's Algorithm)**.** Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then,

$$\exp(At) = \sum_{j=0}^{n-1} r_{j+1}(t) P_j \tag{6.80}$$

where

$$P_0 = I,\ P_j = \prod_{k=1}^{j} (A - \lambda_k I),\ j = 1, \ldots, n \tag{6.81}$$

and $r_1(t), \ldots, r_n(t)$ is the solution of the triangular system

$$r_1' = \lambda_1 r_1 \tag{6.82}$$

$$r_j' = r_{j-1} + \lambda_j r_j,\ j = 2, \ldots, n \tag{6.83}$$

$$r_1(0) = 1,\ r_j(0) = 0,\ j = 2, \ldots, n \tag{6.84}$$

Remark. Each eigenvalue appears in the list repeated according to its multiplicity, and the order of the matrices do not matter since for example $A - \lambda_1 I$ and $A - \lambda_2 I$ commute. Additionally, each r_j can be solved recursively. If $r_1(t)$ is found, $r_2(t)$ can be found by solving a non-homogeneous first order ODE involving $r_1(t)$. We repeat the process until we find $r_1(t), \ldots, r_n(t)$.

Example 23

Find the fundamental matrix for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Solution

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (\lambda-1)^2 = 0$$

Thus $\lambda = 1$ with multiplicity 2. Thus $\lambda_1 = 1,\ \lambda_2 = 1$. Also, $P_0 = I$,

$$P_1 = A - \lambda_1 I = A - I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$P_2 = (A-\lambda_1 I)(A-\lambda_2 I) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Also,

$$r_1' = \lambda_1 r_1 = r_1 \implies \frac{r_1'}{r_1} = 1 \implies \ln(r_1) = t + c \implies r_1 = Ae^t$$

$r_1(0) = 1 \implies A = 1 \implies r_1 = e^t$. For r_2,

$$r_2' = r_1 + \lambda_2 r_2 = e^t + r_2$$

The integrating factor is $\mu = e^{\int -1\,dt} = e^{-t}$. Thus,

$$(\mu r_2)' = t \implies \mu r_2 = t + c \implies r_2 = \frac{t+c}{e^{-t}}$$

$r_2(0) = 0 \implies c = 0 \implies r_2 = t/e^{-t} = te^t$. Thus,

$$\hat{\Phi}(t) = \exp(At) = r_1(t)P_0 + r_2(t)P_1 = e^t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + te^t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

Putzer's algorithm can be applied for all three cases of the eigenvalues of A, including real and distinct eigenvalues and complex conjugate eigenvalues.

Example 24

Find $\hat{\Phi}(t)$ for $A = \begin{pmatrix} 2 & 6 \\ -2 & -4 \end{pmatrix}$.

Solution

$$\det\begin{pmatrix} 2-\lambda & 6 \\ -2 & -4-\lambda \end{pmatrix} = 0 \implies (\lambda+4)(\lambda-2)+12=0 \implies \lambda^2+2\lambda+4=0$$

$$\implies \lambda = \frac{-2\pm\sqrt{4-16}}{2} = \frac{-2\pm\sqrt{12}i}{2} = -1\pm\sqrt{3}i$$

$\therefore \lambda_1 = -1+\sqrt{3}i,\ \lambda_2 = -1-\sqrt{3}i.\ P_0 = I,$

$$P_1 = A-\lambda_1 I = \begin{pmatrix} 2 & 6 \\ -2 & -4 \end{pmatrix} - \begin{pmatrix} -1+\sqrt{3}i & 0 \\ 0 & -1+\sqrt{3}i \end{pmatrix} = \begin{pmatrix} 3-\sqrt{3}i & 6 \\ -2 & -3-\sqrt{3}i \end{pmatrix}$$

$$P_2 = (A-\lambda_1 I)(A-\lambda_2 I) = \begin{pmatrix} 3-\sqrt{3}i & 6 \\ -2 & -3-\sqrt{3}i \end{pmatrix}\begin{pmatrix} 3+\sqrt{3}i & 6 \\ -2 & -3+\sqrt{3}i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

For r_1,

$$r_1' = \lambda_1 r_1 \implies r_1 = e^{\lambda_1 t} = e^{(-1+\sqrt{3}i)t}$$

For r_2,

$$r_2' = r_1 + \lambda_2 r_2 \implies r_2' - (-1-\sqrt{3}i)r_2 = e^{(-1+\sqrt{3}i)t}$$

$\therefore r_2' + (1+\sqrt{3}i)r_2 = e^{(-1+\sqrt{3}i)t}$. The integrating factor is

$$\mu = e^{\int 1+\sqrt{3}i\,dt} = e^{(1+\sqrt{3}i)t}$$

$\therefore (\mu r_2)' = e^{t+\sqrt{3}it-t+\sqrt{3}it} = e^{2\sqrt{3}it}$

$$\therefore \mu r_2 = \int e^{2\sqrt{3}it}\,dt = \frac{1}{2\sqrt{3}i}e^{2\sqrt{3}it} + c$$

$r_2 = \frac{1}{2\sqrt{3}i\mu}e^{2\sqrt{3}it} + \frac{c}{\mu}$. At $r_2(0)=0$,

$$\frac{1}{2\sqrt{3}i} + c = 0 \implies c = -\frac{1}{2\sqrt{3}i}$$

Thus r_2 follows. Then $\hat{\Phi}(t) = r_1(t)P_0 + r_2(t)P_1$.

7 Nonlinear Differential Equations and Stability

7.1 Autonomous Systems and Stability

Consider the two dimensional system

$$x' = f(x, y),\ x(t_0) = x_0 \tag{7.1}$$

$$y' = g(x, y),\ y(t_0) = y_0 \tag{7.2}$$

where f, g are continuous functions that have continuous partial derivatives in some domain D of the xy plane. There is an existence and uniqueness theorem which guarantees the existence of a unique solution

$$x = \phi(t) \tag{7.3}$$

$$y = \psi(t) \tag{7.4}$$

that satisfies the IVP. These are defined in some interval I that contains t_0. The system above is said to be an *autonomous system* since f and g do not depend on the independent variable t but only on the dependent variables x and y. Let us define the following

$$X = (x, y),\ F = (f, g) \quad \text{(a vector valued function)} \tag{7.5}$$

We can write our system as

$$X' = F(X) \tag{7.6}$$

A harmonic oscillator without damping is stable since it always oscillates near the equilibrium with a small perturbation. A damped oscillator is asymptotically stable since after some time, the mass comes to rest at the equilibrium. We denote the length of X as

$$\|X\| = \sqrt{x^2 + y^2} \tag{7.7}$$

The *critical points* of the system are points such that

$$F(X) = 0 \tag{7.8}$$

Critical points represent equilibrium solutions since $X' = 0$. We can now classify the stability of the critical points.

Definition 7.1 (Stable Critical Point)**.** A critical point X^* of the autonomous system is stable if $\forall \epsilon > 0$, $\exists \delta > 0$ such that every solution $X = \Phi(t)$ satisfies

$$\|\Phi(t) - X^*\| < \epsilon \ \forall t > 0 \text{ when } \|\Phi(0) - X^*\| < \delta \tag{7.9}$$

Definition 7.2 (Asymptotically Stable Critical Point)**.** A critical point is asymptotically stable if it is stable and $\exists a > 0$ such that

$$\lim_{t \to \infty} \|\Phi(t) - X^*\| = 0 \text{ when } \|\Phi(0) - X^*\| < a \tag{7.10}$$

Definition 7.3 (Unstable Critical Point)**.** A critical point is unstable if it is not stable.

In essence stability says that if you start close to a point, the solution will stay close for all future points, you can orbit, wobble or bounce near the solution but you never blow away. Asymptotic stability is a stronger condition, it implies stability and the solution eventually goes to the equilibrium as $t \to \infty$.

7.2 Lyapunov Functions

Consider the two dimensional autonomous system

$$x' = f(x, y) \tag{7.11}$$

$$y' = g(x, y) \tag{7.12}$$

Assume the origin $(x, y) = (0, 0)$ is an equilibrium solution. Consider a function $V(x, y)$ defined on a domain $D \subseteq \mathbb{R}^2$ that contains the origin.

Definition 7.4 (Positive Definite)**.** $V(x, y)$ is said to be positive definite on D if

$$V(0,0) = 0, \qquad V(x,y) > 0 \ \forall (x,y) \neq (0,0) \tag{7.13}$$

Definition 7.5 (Negative Definite)**.** $V(x, y)$ is said to be negative definite on D if

$$V(0,0) = 0, \qquad V(x,y) < 0 \ \forall (x,y) \neq (0,0) \tag{7.14}$$

V is said to be positive semidefinite and negative semidefinite respectively if the inequalities are not strict.

Lyapunov's idea is to replace the distance function $\|X\| = \|(x, y)\| = \sqrt{x^2 + y^2}$ by a function $V(x, y)$ called a *Lyapunov function*. The properties of $V(x, y)$ tells us about the stability of the equilibrium at the origin. This is extremely powerful since we do not need to know any information about the autonomous system itself to understand the system's stability; all we have to do is study $V(x, y)$.

Theorem 7.1 (Lyapunov Stability Theorem)**.** Suppose the autonomous system has an isolated critical point at the origin. If there exists a function $V(x, y)$ that is continuous, has continuous first partial derivatives ($\partial V/\partial x,\ \partial V/\partial y$), is positive definite and for which $V' = \frac{d}{dt}V(x(t), y(t))$ is negative definite on some domain D in the xy plane containing the origin, then the origin is an asymptotically stable critical point. If V' is negative semidefinite then the origin is a stable critical point.

Remark. It is useful to think of the Lyapunov function as the energy of the system. If V' is negative definite then $V' < 0$ so the energy keeps decreasing and the solution eventually reaches equilibrium (asymptotic stability).

Theorem 7.2 (Lyapunov Instability Theorem)**.** Suppose the autonomous system has an isolated critical point at the origin. Let V be a function that is continuous and has continuous first partial derivatives. Suppose $V(0,0) = 0$ and that in every neighbourhood of the origin, there is at least one point where V is positive (negative). If there exists a domain D containing the origin such that V' is positive definite (negative definite) on D, then the origin is an unstable critical point.

Remark. If V is positive and V' is positive definite then the energy keeps increasing (doesn't go to 0). Similarly if V is negative and V' is negative definite, then the energy keeps decreasing.

There is no general method to find a Lyapunov function; it is trial and error. Once we find one, we can use the theorems to conclude stability.

Another geometric idea is that

$$V' = \frac{d}{dt}V(x(t), y(t)) = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} \tag{7.15}$$

Recall that for a vector $\mathbf{r} = (x(t), y(t))$, the tangent vector shows the trajectory and is given by

$$T = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \tag{7.16}$$

Then it holds,

$$V' = \nabla V \cdot T = \|\nabla V\|\|T\| \cos\theta \tag{7.17}$$

If $V' \leq 0$ then $\cos\theta \leq 0 \implies \theta \in [\pi/2, 3\pi/2]$. Thus for a given curve $V(x,y) = c$, this shows that the trajectory is either inwards or at worst tangent to the curve. Thus we have stability as the trajectory does not blow away from the solution.

7.3 Competing Species

Let $x(t)$ and $y(t)$ denote the population of two species at time t. Assume the two species compete for a common resource but do not prey on each other. In the absence of one species, the other grows logistically as

$$x'(t) = x(\epsilon_1 - \sigma_1 x) = \epsilon_1 x \left(1 - \frac{x}{\epsilon_1/\sigma_1}\right) \tag{7.18}$$

$$y'(t) = y(\epsilon_2 - \sigma_2 y) = \epsilon_2 y \left(1 - \frac{y}{\epsilon_2/\sigma_2}\right) \tag{7.19}$$

$\epsilon_1, \epsilon_2, \sigma_1, \sigma_2$ are positive constants. ϵ_1 and ϵ_2 denote the growth rate of their respective species while ϵ_1/σ_1 and ϵ_2/σ_2 denote the carrying capacity or saturation level of the species.

We can model competition by adding a dependence of $x(t)$ on y and vice-versa as follows

$$x'(t) = x(\epsilon_1 - \sigma_1 x - \alpha_1 y) \tag{7.20}$$

$$y'(t) = y(\epsilon_2 - \sigma_2 y - \alpha_2 x) \tag{7.21}$$

$\epsilon_1, \epsilon_2, \sigma_1, \sigma_2, \alpha_1, \alpha_2 > 0$. This shows that as y increases, there are less resources for x so $x'(t) < 0$ and the population of x decreases and vice-versa.

The equilibrium points of the above system represent points where x and y coexist. The equilibrium points are given by

$$x(\epsilon_1 - \sigma_1 x - \alpha_1 y) = 0 \tag{7.22}$$

$$y(\epsilon_2 - \sigma_2 y - \alpha_2 x) = 0 \tag{7.23}$$

7.3.0.1 If $x = 0, y = 0$ the equilibrium point is trivial.

$(0, 0)$.

7.3.0.2 If $x \neq 0, y = 0$ then

$$x(\epsilon_1 - \sigma_1 x) = 0 \implies \epsilon_1 - \sigma_1 x = 0 \implies x = \frac{\epsilon_1}{\sigma_1}$$

gives $\left(\frac{\epsilon_1}{\sigma_1}, 0\right)$.

7.3.0.3 If $x = 0, y \neq 0$ then

$$y(\epsilon_2 - \sigma_2 y) = 0 \implies y = \frac{\epsilon_2}{\sigma_2}$$

gives $\left(0, \frac{\epsilon_2}{\sigma_2}\right)$.

7.3.0.4 If $x \neq 0, y \neq 0$ then

$$\epsilon_1 - \sigma_1 x - \alpha_1 y = 0$$

$$\epsilon_2 - \sigma_2 y - \alpha_2 x = 0$$

$$\implies \alpha_1 y + \sigma_1 x = \epsilon_1, \qquad \alpha_2 x + \sigma_2 y = \epsilon_2$$

$$\implies \alpha_1\alpha_2 y + \sigma_1\alpha_2 x = \epsilon_1\alpha_2$$

$$-\sigma_1\alpha_2 x - \sigma_1\sigma_2 y = -\epsilon_2\sigma_1$$

$$\implies y(\alpha_1\alpha_2 - \sigma_1\sigma_2) = \epsilon_1\alpha_2 - \epsilon_2\sigma_1$$

$$\implies y = \frac{\epsilon_1\alpha_2 - \epsilon_2\sigma_1}{\alpha_1\alpha_2 - \sigma_1\sigma_2}$$

$$\implies x = \frac{\epsilon_1\sigma_2 - \alpha_1\epsilon_2}{\alpha_1\alpha_2 - \sigma_1\sigma_2}$$

Thus the last equilibrium point is

$$\left(\frac{\epsilon_1\sigma_2 - \alpha_1\epsilon_2}{\alpha_1\alpha_2 - \sigma_1\sigma_2}, \frac{\epsilon_1\alpha_2 - \epsilon_2\sigma_1}{\alpha_1\alpha_2 - \sigma_1\sigma_2}\right)$$

The equilibrium points are then:

- $(0, 0)$ — trivial
- $(\epsilon_1/\sigma_1, 0)$ — logistic growth of x
- $(0, \epsilon_2/\sigma_2)$ — logistic growth of y
- $\left(\frac{\epsilon_1\sigma_2-\alpha_1\epsilon_2}{\alpha_1\alpha_2-\sigma_1\sigma_2}, \frac{\epsilon_1\alpha_2-\epsilon_2\sigma_1}{\alpha_1\alpha_2-\sigma_1\sigma_2}\right)$ — interior equilibrium

The last equilibrium point is known as the *interior equilibrium* of the system, which can be expressed as the determinants

$$\left(\frac{\begin{vmatrix} \epsilon_1 & \alpha_1 \\ \epsilon_2 & \sigma_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \sigma_1 \\ \sigma_2 & \alpha_2 \end{vmatrix}}, \frac{\begin{vmatrix} \epsilon_1 & \sigma_1 \\ \epsilon_2 & \sigma_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \sigma_1 \\ \sigma_2 & \alpha_2 \end{vmatrix}} \right) \tag{7.24}$$

An interior equilibrium when all the determinants above have the same sign, otherwise either x or y would be negative and negative populations don't make sense.

The special trajectories are lines at $x = 0$ and $y = 0$. If $x = 0$, $x' = 0$ and $y' = \sigma_2 y(\epsilon_2/\sigma_2 - y)$. Thus, trajectories with $x(0) = 0$ stay on the x axis for all $t > 0$. If $0 < y(0) < \epsilon_2/\sigma_2$, the y coordinate increases and if $y(0) > \epsilon_2/\sigma_2$, the y coordinate decreases.

If $y = 0$, $y' = 0$ and $x' = \sigma_1 x(\epsilon_1/\sigma_1 - x)$. Thus, trajectories with $y(0) = 0$ stay on the y axis for all $t > 0$. If $0 < x(0) < \epsilon_1/\sigma_1$, the x coordinate increases and if $x(0) > \epsilon_1/\sigma_1$, the x coordinate decreases.

By the existence and uniqueness theorem, only one solution passes through the special trajectory. This implies once a solution starts in a quadrant, it doesn't leave the quadrant as that would imply passing through a special trajectory.

We can determine the slope of the system using *nullclines.* Nullclines are lines where $x' = 0$ and $y' = 0$ but $x \neq 0,\ y \neq 0$. Define L_1 by the nullcline where $x' = 0$ and L_2 by the nullcline where $y' = 0$,

$$L_1 : \epsilon_1 - \sigma_1 x - \alpha_1 y = 0 \implies y = \frac{\epsilon_1}{\alpha_1} - \frac{\sigma_1}{\alpha_1} x \tag{7.25}$$

$$L_2 : \epsilon_2 - \sigma_2 y - \alpha_2 x = 0 \implies y = \frac{\epsilon_2}{\sigma_2} - \frac{\alpha_2}{\sigma_2} x \tag{7.26}$$

At each nullcline, $x' = 0$ and $y' = 0$ so there is no change in x and y and their slope fields are vertical and horizontal respectively. The nullclines intersect at the interior equilibrium. We can create a table for each region of the nullclines

	Below L_1, Below L_2	Below L_1, Above L_2	Above L_1, Below L_2	Above L_1, Above L_2
x'	$+$	$+$	$-$	$-$
y'	$+$	$-$	$+$	$-$

Interpreting the slope field tells us how our populations change over time.

7.4 Predator-Prey Equations

Consider a prey x that grows logistically in the absence of a predator,

$$x' = rx\left(1 - \frac{x}{K}\right) \tag{7.27}$$

Now assume we have a predator y that dies out exponentially in the absence of prey

$$y' = -cy \tag{7.28}$$

where $c > 0$ is known as the death rate. If the predator and prey interact we introduce a "mass action" term

$$x' = bx - \beta xy \tag{7.29}$$

$$y' = -cy + \gamma xy \tag{7.30}$$

This is the *Lotka-Volterra system.* As population of x, the prey, increases, the predators grow but as the predators grow, the prey go down which causes predators to go down. The equilibrium points are at

$$bx - \beta xy = 0 \implies x(b - \beta y) = 0 \tag{7.31}$$

$$-cy + \gamma xy = 0 \implies y(-c + \gamma x) = 0 \tag{7.32}$$

Solving gives the equilibrium points $(0,0)$ and $(c/\gamma, b/\beta)$. The nullclines are

$$L_1 : y = \frac{b}{\beta}, \qquad L_2 : x = \frac{c}{\gamma} \tag{7.33}$$

At the x-nullcline, $x' = 0$. At $x > c/\gamma$, $\gamma x - c > 0 \implies y' > 0$ ($\uparrow$). At $x < c/\gamma$, $\gamma x - c < 0 \implies y' < 0$ ($\downarrow$). At the y-nullcline, $y' = 0$. At $y > b/\beta$, $b - \beta y < 0 \implies x' < 0$ ($\leftarrow$). At $y < b/\beta$, $b - \beta y > 0 \implies x' > 0$ ($\rightarrow$).

Starting at the y-nullcline we see we have a high predator, low prey population so they die out, this causes the prey population to increase which then increases the predator population due to food abundance and the cycle repeats. The system assumes no carrying capacity for the prey and no saturation in predator consumption which is why the cycle repeats.

For $x > 0,\ y > 0$ (we can't have negative population) we have

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{-cy + \gamma xy}{bx - \beta xy} = \frac{y(-c + \gamma x)}{x(b - \beta y)} \tag{7.34}$$

$$\implies \frac{b - \beta y}{y}\, dy = \frac{-c + \gamma x}{x}\, dx \tag{7.35}$$

Integrating gives

$$b \ln(y) - \beta y = \gamma x - c \ln x + d \implies b \ln(y) - \beta y + c \ln x - \gamma x = d \tag{7.36}$$

The trajectories of the system lie on these level curves; different initial conditions give rise to different values of d. When $x = 0$, $x' = 0$ and $y' = -cy$ so y population reaches 0 as $t \to \infty$. When $y = 0$, $y' = 0$ and $x' = bx$ so x population blows up. The origin then behaves like a saddle; we see above that the interior equilibrium behaves like a centre.

To understand the dynamics around the interior equilibrium $(c/\gamma, b/\beta)$, transform around the origin,

$$u = x - \frac{c}{\gamma} \implies u' = x' = bx - \beta xy \tag{7.37}$$

$$v = y - \frac{b}{\beta} \implies v' = y' = -cy + \gamma xy \tag{7.38}$$

Substitute,

$$u' = b\left(u + \frac{c}{\gamma}\right) - \beta\left(u + \frac{c}{\gamma}\right)\left(v + \frac{b}{\beta}\right) \tag{7.39}$$

$$= bu + \frac{bc}{\gamma} - \left(u + \frac{c}{\gamma}\right)(\beta v + b) \tag{7.40}$$

$$= bu + \frac{bc}{\gamma} - \left(\beta uv + bu + \frac{\beta cv}{\gamma} + \frac{bc}{\gamma}\right) \tag{7.41}$$

$$= bu + \frac{bc}{\gamma} - \beta uv - bu - \frac{\beta cv}{\gamma} - \frac{bc}{\gamma} \tag{7.42}$$

$$= -\beta uv - \frac{\beta cv}{\gamma} \tag{7.43}$$

Substitute,

$$v' = -c\left(v + \frac{b}{\beta}\right) + \gamma\left(u + \frac{c}{\gamma}\right)\left(v + \frac{b}{\beta}\right) \tag{7.44}$$

$$= -cv - \frac{bc}{\beta} + (\gamma u + c)\left(v + \frac{b}{\beta}\right) \tag{7.45}$$

$$= -cv - \frac{bc}{\beta} + \gamma uv + \frac{b\gamma u}{\beta} + cv + \frac{bc}{\beta} \tag{7.46}$$

$$= \gamma uv + \frac{b\gamma u}{\beta} \tag{7.47}$$

Thus,

$$u' = -\beta uv - \frac{\beta c}{\gamma} v \tag{7.48}$$

$$v' = \gamma uv + \frac{\gamma b}{\beta} u \tag{7.49}$$

Near the interior equilibrium $(c/\gamma, b/\beta)$ in (x, y) we are near $(0, 0)$ in the transformed system. Near the interior equilibrium, we can avoid βuv and γuv since they are really small to get

$$u' = -\frac{\beta c}{\gamma} v \tag{7.50}$$

$$v' = \frac{\gamma b}{\beta} u \tag{7.51}$$

Thus,

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & -\beta c/\gamma \\ \gamma b/\beta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{7.52}$$

The eigenvalues to this matrix are given by

$$\det \begin{pmatrix} -\lambda & -\beta c/\gamma \\ \gamma b/\beta & -\lambda \end{pmatrix} = 0 \implies \lambda^2 + bc = 0 \implies \lambda = \pm i\sqrt{bc} \tag{7.53}$$

These are complex (no real parts) so we have a centre. Now our system is once again

$$u' = -\frac{\beta c}{\gamma} v \tag{7.54}$$

$$v' = \frac{\gamma b}{\beta} u \tag{7.55}$$

This satisfies

$$\frac{u^2}{(c\kappa/\gamma)^2} + \frac{v^2}{(b\kappa/\beta\sqrt{c/b})^2} = 1 \tag{7.56}$$

for some $\kappa > 0$. The above is an ellipse with period

$$\frac{2\pi}{\sqrt{bc}} \tag{7.57}$$

In x, y the system close to the interior equilibrium behaves as

$$x = \frac{c}{\gamma} + u = \frac{c}{\gamma} + \frac{\kappa c}{\gamma}\cos(\sqrt{bc}t + \theta) \tag{7.58}$$

$$y = \frac{b}{\beta} + v = \frac{b}{\beta} + \kappa\frac{b}{\beta}\sqrt{\frac{c}{b}}\sin(\sqrt{bc}t + \theta) \tag{7.59}$$

κ determines how big the ellipse and θ determines where on the ellipse we start depending on the initial conditions.

Remark. The predator and prey population sizes vary with period $2\pi/\sqrt{bc}$ which do not depend on the IC; it varies by b (growth rate of prey) and $\sqrt{bc}$ (death rate of predator); large rates cause period to be small. The prey and predator populations are out of phase by a quarter of a cycle ($\pi/2\sqrt{bc}$); the prey leads and predator lags. The amplitudes of the oscillations depend on the initial conditions (through κ) and the parameters of the problem.